STUDIES ON SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS

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Doctor of Philosophy

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DECLARATION

I, hereby declare that the work included in this thesis entitled " STUDIES ON SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS " is carried out by me under the guidance and supervision of Dr. Jagdish A. Nanware, Department of PG Studies and Research in Mathematics, Shrikrishna Mahavidyalaya, Gunjoti. The work is original and has not been submitted is part or in full to any other University or Institute for award of any research degree. The extent of information derived from existing literature has been indicated in the body of the thesis at appropriate places giving the references.

Place: Gunjoti

Date: 24^{th} April 2022

Research Scholar

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CERTIFICATE

This is to certify that work embodied in the thesis entitled, " STUDIES ON SOLUTIONS OF FRACTIONAL DIFFERENTIAL EQUATIONS AND APPLICATIONS" being submitted by Mrs.Madhuri Nivrutti Gadsing to the Dr.Babasaheb Ambedkar Marathwada University, Aurangabad for the award of degree of Doctor of Philosophy in Mathematics under the Faculty of Science and Technology, is a record of bonafide research work carried out by her under my guidance and supervision, and has fulfilled the requirements for the submission of this thesis to my knowledge, has reached requisite standard. The results contained in this thesis have not been submitted in part or in full, to any other University or institute for the award of any degree or diploma.

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Mrs.Madhuri Nivrutti Gadsing

Contents

Dedicated to...

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Mother Mrs.Sujata and
Father Mr.Nivrutti Gadsing

Chapter 1

Introduction

1.1 Historical Development and Literature Review

The beginning of the fractional calculus is considered to be the Leibniz's letter to L'Hospital in 1695, where they discussed for differentiation of non-integer order. Leibniz wrote to L'Hospital: ″*... This is an apparent paradox from which, one day, useful consequences will be drawn*″. Up to the middle of 20th century, a remarkable contributions to the theory of fractional calculus by famous mathematician are known : Euler (1730), Lagrange (1772), Laplace (1812), J.B.J. Fourier (1822), N. Abel (1823-1826), J. Liouville (1832-1837), B. Riemann (1847), K. Grunwald (1867-1872), A. Letnikov (1868-1872), Heaviside (1892-1912), A. Marchaud (1973-1999), A. Erdelyi (1930-1977), H. Weyl (1917) etc.

The first work, devoted to fractional calculus, is the book by Oldham and Spanier published in 1974. By the second half of the twentieth century the first specialized conference concerned solely with the theory and applications on fractional calculus in 1974 was held at the University of New Haven USA and more than 2000 papers have been published. Letnikov (1868), S. G. Samko (1993), Podlubny (1999), A. A. Kilbas (2006), O. I. Marichev, V. S. Kiryakova, K. S. Miller and B. Ross and more recently Lakshmikantham, S. Abbas,

M. Benchohra, J.R.Greuf, Zhou, Das, Ray, Herrmann and more are devoted to the classical period of the development of fractional calculus. The first issue of the mathematical journal 'Fractional Calculus and Applied Analysis' was printed in 1998. This journal is solely concerned with topics on the theory of fractional calculus and its applications. Finally, the large conference concerned solely with 'Fractional Differentiation and its Applications' was held in Bordeaux in 2004, where no less than 104 talks were given in the field of fractional calculus.

Now a days, fractional calculus has attracted much more attention as it plays an important role in the field of science and engineering in which the study on stability of fractional differential equations is important. Fractional differential equation is a generalization of ordinary differential equation of fractional order. Many mathematical models of real problems arising in various fields of science and engineering are either linear or nonlinear systems. Recently, stability of fractional differential equations and the analytical methods of solutions of linear fractional differential equations, nonlinear fractional differential equations. Fractional differential equations represent an important tool in science, engineering, technology and economics and applications included population models, control engineering, electrical network analysis, gravity and medicine etc.[27]. Most of the researchers are attracted towards fractional differential equations as many phenomena in various branches of science and engineering are modeled. Many applications are found in control systems, visco-elasticity, electrochemistry, pharmacokinetics, food science etc.[55]. Significant contributions by researchers have been recorded in the monograph of Kilbas et al.[30]. Some results on the theory of fractional differential equations due to Lakshmikantham et al. can be seen in [32, 35, 37, 49, 62, 64, 77]. The study of theory of fractional differential equations [31, 32, 35] parallel to the well-known theory of ordinary differential equations [31, 33] has been growing independently. The existence of solutions of Riemann-Liouville fractional differential equations and uniqueness of solutions was proved by Lakshmikantham and Vatsala [31, 32].

The earliest study on stability of fractional differential equations started by D. Matignon [39]. The recent study of linear fractional differential equations with Riemann-Liouville derivative and the same fractional order α, where $0 < \alpha < 1$ is studied by Qian et al.[56]. In recent years, many effective methods for obtaining approximations or numerical solutions of fractional differential equations have been presented. These methods includes fixed point technique [3, 26, 38], Monotone iterative technique [31, 35, 55], Generalized monotone iterative method [22], Adomian Decomposition Method , Power Series Method [55], the reproducing Kernel method [69] and the Wavelet method . Except these methods, there are some classical solution techniques too, e.g. Laplace and Fourier transform method [19], Green's function method, Maximum principle [24, 61] Mellin transform method and Method of orthogonal polynomials [29, 41, 55].

The monotone iterative technique [14] is very useful for the investigation of theoretical as well as constructive results in the sector. McRae [40] developed monotone method for Riemann-Liouville fractional differential equations with initial conditions and studied the qualitative properties of solutions of initial value problem. Existence and uniqueness of solution of Riemann-Liouville fractional differential equations, Caputo fractional differential equations with various conditions such as integral boundary conditions [42–44, 46–48, 50], initial value problems [45, 51, 71], periodic boundary value problems [22, 23, 25, 70, 72] is obtained by Nanware et al. and others [75] and developed monotone method. Existence of positive solutions for nonlinear fractional differential equations with boundary condition and integral boundary conditions is also obtained by researchers [52, 54, 63, 68].

In the recent years, the theory of singular boundary value problems has become an important area of investigation [2, 4, 5, 12, 16–18, 67, 74]. They got the existence of solutions by using various methods such as monotone iterative technique coupled with lower and upper solution and fixed point theorem. By using monotone iterative method X. Zhang et al. [76] obtained the existence and uniqueness of positive solutions for a class of higher conjugate-type fractional differential equations with one nonlocal term. In recent paper, S. Song et al.[60] investigated the existence of extremal solutions for Riemann-Liouville fractional differential equations with integral boundary conditions.

In 2016, Almeida [8] introduced ψ-Caputo fractional differential operator. For more details see [3, 6, 7, 20], and the references given therein. Derbazi et al. [20, 21] developed monotone iterative technique to study the existence and uniqueness of solution for initial value problem of nonlinear fractional differential equations including ψ-Caputo derivative.

Wang et al.[66] in 2012 proved existence results for nonlinear system of Riemann-Liouville fractional differential equations and developed monotone iterative technique. Recently, Nanware et al. [23, 47, 48, 50] and Dhaigude et al. [23] obtained existence and uniqueness results of the system of fractional differential equations with various type of conditions involving Riemann-Liouville fractional derivative and Caputo fractional derivative of order μ;

$0 < \mu < 1$ using monotone iterative technique. Wei et al. [71] and Nanware et al. [45] proved existence results for system of initial value problems for fractional differential equations involving Riemann-Liouville sequential fractional derivative via monotone iterative technique.

At the present day, different kinds of fixed point theorems are widely used to prove the existence and uniqueness of solutions for various classes of nonlinear fractional differential equations. Recently, Nanware [50] obtained existence results using Banach and Schaufer's fixed point theorem and Leray-Schauder type nonlinear alternative. By using contraction mapping principle, existence and uniqueness of solutions for system of nonlinear implicit fractional differential equations was studied by Nanware [50]. Matar [38] proved existence of positive solution for initial value problem of nonlinear fractional differential equations by the method of upper and lower solutions and using Schauder and Banach fixed point theorems [9]. Many researchers investigated the positive solutions for nonlinear fractional differential equations by the method of upper and lower solutions and using Krasnoselskii and Banach fixed point theorems [1, 10, 13, 54, 63].

1.2 Basic Results

In this section we start with Euler's integrals such as Beta, Gamma and Mittag-Leffler functions which plays important role in the development of fractional calculus. First we consider Beta and Gamma functions.

Definition 1.1. [55] The first Eulerian integral called Beta function denoted by $\beta(p,q)$ and defined by means of the integral $\beta(p,q) = \int_0^1 r^{(p-1)}(1-r)^{(q-1)}dr$ for $(\mathscr{R}(p) > 0)$, $(\mathscr{R}(q) > 0)$, where $\mathscr{R}(p)$ is real part of p.

Definition 1.2. [55] The second Eulerian integral is called Gamma function denoted by $\Gamma(r)$ and is defined by means of the integral $\Gamma(r) = \int_0^\infty e^{-s}s^{r-1}ds, \quad (\mathscr{R}(s) > 0)\ s \in C.$

The relation between Beta and Gamma functions is given by

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = \beta(q,p).$$

Mittag-Leffler function is a generalization of exponential function. One parameter and two parameter Mittag-Leffler functions are defined as follows:

Definition 1.3. [55] Mittag-Leffler function of one parameter denoted by $E_\mu(r)$ is defined as

$$E_\mu(r) = \sum_{j=0}^{\infty} \frac{r^j}{\Gamma(\mu j + 1)}$$

1. When $\mu = 0$,

$$E_0(r) = \sum_{j=0}^{\infty} \frac{r^j}{\Gamma(1)} = 1 + r + r^2 + \ldots = \frac{1}{1-r}, \qquad |r| < 1.$$

2. For $\mu = 1$ and 2, the one parameter Mittag-Leffler function gives the function e^r and $\cosh r$ respectively.

Definition 1.4. [55] Mittag-Leffler function of two parameter denoted by $E_{\mu,\nu}(r)$ is defined as

$$E_{\mu,\nu}(r) = \sum_{j=0}^{\infty} \frac{r^j}{\Gamma(\mu j + \nu)}, \quad (\mu > 0, \nu > 0).$$

We start with the definitions of Riemann-Liouville fractional integrals and Riemann-Liouville fractional derivatives. Then we introduce the definitions of Caputo fractional derivatives and $\psi-$Caputo fractional derivatives. We also consider basic properties of these fractional integrals and fractional derivatives.

Let $\mu \in \mathbb{R}_+$ and $n = [\mu]$, where $[.]$ is the greatest integer function. Assume that $u(r)$ is continuous on an interval $[a, b]$.

Definition 1.5. [55] The Riemann-Liouville fractional integral of a function $u(r)$ of order μ is denoted by $I^{\mu}_{a+}u(r)$ and is defined as

$$I^{\mu}_{a+}u(r) = \frac{1}{\Gamma(\mu)}\int_{a+}^{r}(r-\tau)^{\mu-1}u(\tau)d\tau.$$

Now, we discuss useful properties of Riemann-Liouville fractional integrals.

Property I [55]. Suppose $u(r)$ has the continuous derivatives for $r \geq 0$ then

$$\lim_{\mu\to 0} I^{\mu}_{0}u(r) = u(r).$$

Property II [55].[Additive index property or Semigroup property]

$$I^{\mu}_{a+}I^{\nu}_{a+}u(r) = I^{\mu+\nu}_{a}u(r).$$

We define Riemann-Liouville fractional derivative of order μ and discuss some useful properties of Riemann-Liouville fractional derivative.

Definition 1.6. [55] The Riemann-Liouville fractional derivative of $u(r)$ of order μ is denoted by $D^{\mu}_{a}u(r)$ on a finite interval $[a, b]$, subset of real axis $\mathbb{R}$ is defined as

$$D^{\mu}_{a+}u(r) = \frac{1}{\Gamma(k-\mu)}\frac{d^k}{dr^k}\int_{a}^{r}(r-\tau)^{k-\mu-1}u(\tau)d\tau, \qquad (k-1 \leq \mu < k).$$

For $k = 1$, we have

$$D^{\mu}_{a+}u(r) = \frac{1}{\Gamma(1-\mu)}\frac{d}{dr}\int_a^r (r-\tau)^{-\mu}u(\tau)d\tau, \qquad (0 < \mu < 1),$$

where $u(r) \in C[a, b]$ and $a < r < b$.

Now, we consider the properties of Riemann-Liouville fractional derivatives required in further discussion.

Property I [55]. [Linearity] Riemann-Liouville fractional derivative operator is linear. That is

$$D^{\mu}_{a+}\left[\sigma_1 u_1(r) + \sigma_2 u_2(r)\right] = \sigma_1 D^{\mu}_{a+}\left[u_1(r)\right] + \sigma_2 D^{\mu}_{a+}\left[u_2(r)\right],$$

where D^{μ}_{a+} denotes the Riemann-Liouville fractional derivative.

Property II [55]. Riemann-Liouville fractional derivative is the left inverse to the Riemann-Liouville fractional integral of the same order μ. That is for $\mu > 0$ and $r > a$,

$$D^{\mu}_{a+}I^{\mu}_{a+}u(r) = u(r).$$

Property III [55]. If u(r) is continuous and if $\mu \geq \nu \geq 0$, then the derivative $D^{\mu-\nu}_{a+}u(r)$ exists.

Next, we consider Caputo fractional derivative and introduce some properties of the Caputo derivative.

Definition 1.7. [30] The Caputo fractional derivative of order $\mu > 0$ of a function $u : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$^{c}D^{\mu}u(r) = \frac{1}{\Gamma(n-\mu)} \int_0^r (r-s)^{n-\mu-1} u^{(n)}(s)\, ds.$$

where $n = [\mu] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Remark 1.2.1.
1. In general Riemann-Liouville fractional operator and Caputo fractional operator do not coincide $D^{\mu}u(r) \neq {}^{c}D^{\mu}u(r)$.

2. The Caputo fractional derivative of the constant function is zero.

3. For the Caputo fractional derivative

$$^{c}D^{\mu}\, {}^{c}D^{m}u(r) = {}^{c}D^{\mu+m}u(r)$$

where $(m = 0, 1, 2, \dots (n-1), \quad n-1 < \mu < n)$, while for the Riemann-Liouville fractional derivative $D^{m}D^{\mu}u(r) = D^{\mu+m}u(r)$ where $(m = 0, 1, 2, \dots (n-1), \quad n-1 < \mu < n)$.

We now introduce some important properties of Caputo derivative required in further discussion.

Property I [30]. [Linearity] Caputo fractional derivative operator is linear. That is

$$ {}^{c}D^{\mu}_{a+}\left[\sigma_1 u_1(r)+\sigma_2 u_2(r)\right]=\sigma_1 {}^{c}D^{\mu}_{a+}\left[u_1(r)\right]+\sigma_2 {}^{c}D^{\mu}_{a+}\left[u_2(r)\right] $$

where ${}^{c}D^{\mu}_{a+}$ denotes the Caputo fractional derivative.

Property II [30]. [Non-commutation] Let $n-1<\mu<n$, $\mu\in\mathbb{R}$ $m,n\in\mathbb{N}$ and the function $u(r)$ is such that ${}^{c}D^{\mu}_{0}u(r)$ exists. Then

$$ {}^{c}D^{\mu}_{0}\,D^{m}_{0}u(r)={}^{c}D^{\mu+m}_{0}u(r)\neq D^{m}\;{}^{c}D^{\mu}_{0}u(r) $$

. In general the two operators, Riemann- Liouville and Caputo do not coincide, but if function $u(r)$ be such that $u^{(k)}(0)=0, k=0,1,2,\ldots,n-1$, then the Riemann-Liouville and Caputo fractional derivatives coincides ${}^{c}D^{\mu}_{0}u(r)=D^{m}_{0}u(r)$.

Property III [55]. Let $r>0$, $n-1<\mu<n$, $n\in Z^{+}$, then the following relation between Riemann-Liouville and Caputo fractional derivatives holds

$$ {}^{c}D^{\mu}u(r)=D^{\mu}u(r)-\sum_{j=0}^{n-1}\frac{r^{j}}{j!}u^{j}(0). $$

Property IV [30].[Composition with integer order derivative]

$$ {}^{c}D^{\mu}({}^{c}D^{m}u(r))={}^{c}D^{m}({}^{c}D^{\mu}u(r))={}^{c}D^{\mu+m}u(r) $$

where $f^{(k)}(0)=0, k=n,n+1,...,m\,(m=0,1,2,...;n-1<\mu<n)$.

We present and study some properties of a ψ-Caputo fractional derivative with respect to another function. The idea is to combine the definition of Caputo fractional derivative with the Riemann-Liouville fractional derivative with respect to another function. Also we study the main properties of ψ-Caputo operator.

Let $J = [a, b]$, where $0 \leq a < b < \infty$, be a finite interval and $\psi : J \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(r) \neq 0$, for all $r \in J$.

Definition 1.8. [8]. The left-sided ψ-Riemann-Liouville fractional integral of order $\mu > 0$ for an integrable function $z : J \to \mathbb{R}$ with respect to function ψ is defined as follows

$$I_{a^+}^{\mu,\psi} z(r) = \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1} z(s)\, ds,$$

where $\psi'(r) \neq 0$ and $\Gamma(.)$ is the gamma function.

Definition 1.9. [8]. Let $n \in \mathbb{N}$ and $\psi, z \in C^n(J, \mathbb{R})$ be two functions. The left-sided ψ-Riemann-Liouville fractional derivative of function z of order $n-1 < \mu < n$ with respect to another function ψ is defined by

$$\begin{aligned} D_{a^+}^{\mu,\psi} z(r) &= \left(\frac{1}{\psi'(r)} \frac{d}{dr} \right)^n I_{a^+}^{n-\mu,\psi} z(r) \\ &= \frac{1}{\Gamma(n-\mu)} \left(\frac{1}{\psi'(r)} \frac{d}{dr} \right)^n \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{n-\mu-1} z(s)\, ds, \end{aligned}$$

where $n = [\mu] + 1$ and $[\mu]$ denotes the integer part of real number μ.

Remark 1.2.2. Some well known fractional derivatives are the particular cases of the ψ-Caputo fractional derivative. That is, if we consider $\psi(r) = r, \log r$ and r^σ, respectively we obtain the Riemann-Liouville fractional derivative, the Caputo-Hadamard fractional derivative and the Caputo-Erdelyi-Kober fractional derivative respectively [58].

Definition 1.10. [8]. Let $n \in \mathbb{N}$ and let $\psi, z \in C^n(J, \mathbb{R})$ be two functions. The left-sided ψ-Caputo fractional derivative of z of order $n-1 < \mu < n$ with respect to another function ψ is defined by

$$^cD_{a^+}^{\mu,\psi} z(r) = I_{a^+}^{n-\mu,\psi} \left(\frac{1}{\psi'(r)} \frac{d}{dr} \right)^n z(r),$$

where $\psi'(r) \neq 0$ and $n = [\mu] + 1$ for $\mu \notin \mathbb{N}$, $n = \mu$ for $\mu \in \mathbb{N}$. Symbolically, we denote

$$z_\psi^{[n]}(r) = \left(\frac{1}{\psi'(r)} \frac{d}{dr} \right)^n z(r).$$

From the definition, it is clear that

$$^cD_{a^+}^{\mu,\psi} z(r) = \begin{cases} \frac{1}{\Gamma(n-\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{n-\mu-1} z_\psi^{[n]}(s)\, ds, & \text{if } \mu \notin \mathbb{N} \\ z_\psi^{[n]}(r), & \text{if } \mu \in \mathbb{N}. \end{cases} \tag{1.2.1}$$

In particular, when $\mu \in (0,1)$, we have

$$^{c}D_{a^+}^{\mu,\psi} z(r) = \frac{1}{\Gamma(1-\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{-\mu} z_{\psi}^{[n]}(s)\, ds.$$

Observe that ψ-Caputo derivative of a constant function is zero.

Remark 1.2.3. If $z \in C^n(J,\mathbb{R})$, the ψ-Caputo fractional derivative of $z(r)$ of order μ is defined in terms of left-sided ψ-Riemann-Liouville fractional derivative as

$$^{c}D_{a^+}^{\mu,\psi} z(r) = D_{a^+}^{\mu,\psi} z(r) - \sum_{k=0}^{n-1} \frac{z_{\psi}^{[k]}(a)}{k!}[\psi(r)-\psi(a)]^k.$$

We introduce some important properties of ψ-Caputo fractional derivatives required in further study.

Property I [8]. [Semigroup property] Let $\mu, \nu > 0$, $z \in C(J,\mathbb{R})$. Then

$$I^{\mu,\psi} I^{\nu,\psi} z(r) = I^{\mu+\nu,\psi} z(r),\ r \in J.$$

Property II [8]. Let $z : J \to \mathbb{R}$ and $\mu > 0$. The following holds:

1. If $z(r) \in C(J,\mathbb{R})$ then

$$^{c}D_{a^+}^{\mu,\psi} I^{\mu,\psi} z(r) = z(r),\ r \in J.$$

The ψ-Caputo fractional derivative is an inverse operation for the Riemann-Liouville fractional integral.

2. If $z \in C^{n-1}(J, \mathbb{R})$, $n-1 < \mu < n$, then

$$I_{a^+}^{\mu,\psi} {}^c D_{a^+}^{\mu,\psi} z(r) = z(r) - \sum_{k=0}^{n-1} \frac{z_\psi^{[k]}(a)}{k!}[\psi(r) - \psi(a)]^k, \quad r \in J.$$

Property III [8]. For $r > a$, $\mu \geq 0$ and $\nu > 0$, we have

(i) $I_{a^+}^{\mu,\psi}[\psi(r) - \psi(a)]^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)}[\psi(r) - \psi(a)]^{\nu+\mu-1}$,

(ii) ${}^c D_{a^+}^{\mu,\psi}[\psi(r) - \psi(a)]^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)}[\psi(r) - \psi(a)]^{\nu-\mu-1}$,

(iii) ${}^c D_{a^+}^{\mu,\psi}[\psi(r) - \psi(a)]^k = 0$, for all $k \in \{0, 1, \ldots, n-1\}$, $n \in \mathbb{N}$.

Note that through out the thesis we shall denote Riemann-Liouville fractional integrals and Riemann-Liouville fractional derivatives by the notations I^μ and D^μ respectively. Also, we denote Caputo fractional derivative by the notation ${}^c D^\mu$ and ψ-Caputo fractional derivative by the notation ${}^c D^{\mu,\psi}$.

Finally we consider following basic results for further study.

Theorem 1.11. *[15](Ascoli-Arzela Theoerm). If a sequence $\{u_n\}_0^\infty$ in $C(X)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*

Theorem 1.12. *[11] (Lebesgue dominated convergence theorem). Suppose $f_n : \mathbb{R} \to \mathbb{R}$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume there is an integrable $g : \mathbb{R} \to [0, \infty)$ with $|f_n(x)| \leq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and*

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n d\mu = \int_{\mathbb{R}} \lim_{n\to\infty} f_n d\mu = \int_{\mathbb{R}} f d\mu$$

Theorem 1.13. *[59]. Let E be a nonempty closed convex subset of a Banach space B and $\Phi : E \to E$ be a contraction operator. Then there is a unique $w \in E$ with $\Phi w = w$.*

Theorem 1.14. *[59](Krasnoselskii fixed point theorem). Let E be a nonempty closed convex subset of a Banach space B and let P and Q two operators defined on E with values in B such that $Pu+Qv \in E$, for every pair $u, v \in E$, the operator P is completely continuous and the operator Q is a contraction.Then there exist $w \in E$ such that $w = Pw + Qw$.*

Theorem 1.15. *[57] (Dini's Theorem). Let B be a compact metric space. Let $g : B \to \mathbb{R}$ be a continuous function and $g_k : B \to \mathbb{R}$, $n \in \mathbb{N}$ be a sequence of continuous functions. If $\{g_k\}$ converge pointwise to f and if $g_k(r) \geq g_{k+1}(r)$ for all $r \in B$ and all $k \in \mathbb{N}$ then $\{g_k\}$ converges uniformly to g.*

Theorem 1.16. *[27](Weissinger's fixed point theorem). Assume (A, ρ) to be a non empty complete metric space and let θ_i for every $i \in \mathbb{N}$ such that $\sum_{i=0}^{\infty} \theta_i$ converges. Furthermore, let the mapping $T : A \to A$ satisfy the inequality $\rho(T^i x, T^i y) \leq \theta_i \rho(x, y)$, for every $i \in \mathbb{N}$ and every $x, y \in A$. Then T has a unique fixed point x^*. Moreover, for any $x_0 \in A$, the sequence $\{T^i x_0\}_{i=1}^{\infty}$ converges to fixed point x^*.*

Theorem 1.17. *[30]. Let $\mu > 0$, $n \in \mathbb{N}$ and $n = [\mu]$. Moreover, let $k > 0$, $h^* > 0$ and $a_0 \in \mathbb{R}$. Define $H : [0, h^*] \times [a_0 - k, a_0 + k]$ and let the function $f : H \to \mathbb{R}$ be continuous. Then for some $h > 0$,the function $v(r) \in C(0, h]$ is a solution to the fractional differential equation of Riemann-Liouville type*

$$D^{\mu} v(r) = f(r, v(r)) \quad v(r_0) = v_0 \tag{1.2.2}$$

if and only if it is a solution of the Volterra fractional integral equation

$$v(r) = v_0 + \frac{1}{\Gamma(\mu)} \int_0^r (r - s)^{\mu - 1} f(s, v(s)ds). \tag{1.2.3}$$

Now, consider the following nonlinear differential equation of order $\mu > 0$

$$[^c D^{\mu} u](y) = f(y, u(y))$$

where ${}^{c}D^{\mu}u(y)$ is Caputo fractional derivative. The corresponding Volterra integral equation is given by

$$u(y) = \sum_{j=0}^{n-1} \frac{a_j (y-a)^j}{j!} + \frac{1}{\Gamma(\mu)} \int_a^y \frac{f(r, u(r))}{(y-r)^{1-\mu}} dr, \ (0 \leq y \leq 1).$$

Theorem 1.18. *[22]. Let $m \in C_p([r_0, T], \mathbb{R})$. Suppose that for any $r_1 \in [r_0, T]$, we have $m(r_1) = 0$ and $m(r) < 0$ for $r_0 \leq r < r_1$, then it follows that $D^q m(r_1) \geq 0$.*

Theorem 1.19. *[34]. Let $\{u_\epsilon(r)\}$ be a family of continuous functions on $[r_0, T]$, for each $\epsilon > 0$ where $D^\mu u_\epsilon(r) = f(r; u_\epsilon(r))$; $u_\epsilon(c) = u_\epsilon(r)$ and $|f(r; u_\epsilon(r))| \leq M$ for $r_0 \leq r \leq T$. Then the family $\{u_\epsilon(r)\}$ is equicontinuous on $[r_0, T]$.*

Lemma 1.20. *[3] If $\mu > 0$ and $z, \psi \in C[a, b]$, then*

(a) *$I_{a^+}^{\mu,\psi}(.)$ is linear and bounded from $C[a, b]$ to $C[a, b]$.*

(b) *$I_{a^+}^{\mu,\psi} z(a) = \lim_{r \to a} I_{a^+}^{\mu,\psi} z(r) = 0$.*

1.3 The purpose of Thesis

The purpose of thesis is to study existence and uniqueness of solutions of fractional differential equations with various conditions by using fixed point theorem and developing monotone iterative method involving Riemann-Liouville, Caputo, ψ-Caputo fractional derivatives. Keeping in the mind and using the beauty of fixed point theorems like Krasnoselskii's and Banach fixed point theorem, existence and uniqueness of solutions will be obtained. Moreover, attempt has been made to develop the monotone iterative method for the system of fractional differential equations involving Riemann-Liouville, ψ-Caputo fractional derivatives. It is extended to higher order singular nonlinear conjugate type fractional differential equations. The method of lower and upper solution is developed. As an application of monotone iterative method it is generalized to ψ-Caputo fractional derivatives. In all, existence and uniqueness result of solutions of fractional differential equations and system of fractional differential equations under different conditions will be proved.

1.4 Organization of the Thesis

The organization of the thesis is as follows:

Chapter 1 is introductory. Special functions, Riemann-Liouville fractional integral and derivative, Caputo derivative and ψ-Caputo derivatives are defined. The properties of these derivatives are considered. The basic results are also given.

Chapter 2 deals to investigate the existence of solution of nonlinear Liouville-Caputo fractional differential equations of order $1 < \mu \leq 2$. Existence and uniqueness of a solution are investigated by applying Krasnoselskii and Banach fixed point theorems and the method of lower and upper solutions.

Chapter 3 deals with the monotone iterative technique for higher order singular nonlinear $(l-1, 1)$ conjugate-type fractional differential equation with one nonlocal term. By deriving properties of the Green's function and using the fixed point theorem, we have proved existence result. Existence and uniqueness of solutions of the problem are obtained.

Chapter 4 deals with the monotone iterative technique for nonlinear boundary value problems involving ψ-Caputo fractional derivative. Monotone iterative technique combined with coupled lower-upper solutions is developed for nonlinear problem and qualitative properties of solutions such as existence-uniqueness are obtained.

Chapter 5 deals with nonlinear system of initial value problems involving Riemann-Liouville fractional derivative when the functions $f_i, (i = 1, 2)$ are quasimonotone nondecreasing (nonincreasing). Monotone iterative technique coupled with lower and upper solutions is developed and successfully applied to study qualitative properties of solutions.

Chapter 6 deals with the monotone iterative technique for coupled system of initial value problem involving ψ-Caputo fractional derivative. We apply the comparison result and coupled lower-upper solutions to develop monotone technique for the initial value problem. Minimal and maximal solutions are obtained by using developed monotone technique.

Chapter 7 deals with the study qualitative properties such as existence-uniqueness of solutions for nonlinear boundary value problems involving ψ-Caputo fractional derivative. Monotone iterative technique is also developed.

Conclusion of each chapter is given at the end of chapter. The list of references is added in alphabetical order at the end of the thesis.

Chapter 2

Nonlinear Liouville-Caputo Fractional Differential Equations

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2.1 Introduction

In recent years, theory of fractional differential equations has become an important investigation area.(Kilbas et al. [30], I. Podlubny [55], and Samko et al. [58]). The basic theory for initial value problems for fractional differential equations involving the Riemann-Liouville and Liouville-Caputo differential operator was discussed by Diethelm [28]. Many interesting results of the existence of solutions of various classes of fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives with the initial condition, the integral boundary conditions have been studied extensively by several researchers (see [3, 32–34, 36, 47, 48, 50, 53, 73, 75] and the references therein).

Recently, Matar [38] used the method of upper and lower solutions and Schauder and Banach fixed point theorems to obtain the existence and uniqueness of positive solution for the nonlinear fractional differential equations

$$\begin{cases} {}^cD^{\mu}w(r) = f(r, w(r)), & 0 < r \leq 1, \\ w(0) = 0, \quad w^{'}(0) = \zeta > 0, \end{cases}$$

where ${}^cD^{\mu}$ is the standard Caputo fractional derivative of orderμ, $1 < \mu \leq 2$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous function.

By employing the upper and lower solutions and Schauder and Banach fixed point theorems, Boulares et al. [13] investigated existence and uniqueness of positive solutions for the nonlinear fractional differential equations

$$\begin{cases} {}^cD^{\mu}w(r) = f(r, w(r)) + {}^cD^{\mu-1}h(r, w(r)), \quad 0 < r \leq T, \\ w(0) = \zeta_1, \quad w^{'}(0) = \zeta_2 > 0, \end{cases}$$

where ${}^cD^{\mu}$ is the standard Liouville-Caputo's fractional derivative of order $\mu, 1 < \mu \leq 2$, $h, f : [0, T] \times [0, \infty) \to [0, \infty)$ are given continuous function, h is nondecreasing in w and $\zeta_2 \geq h(0, \zeta_1)$. Ardjouni et al. [9] studied the existence and uniqueness of positive solutions for the first-order nonlinear Caputo-Hadamard fractional differential equations. Also Ardjouni et al. [10] studied the existence and uniqueness of positive solutions for the first-order nonlinear Liouville - Caputo fractional differential equations by using the upper and lower solutions and use Krasnoselskii and Banach fixed point theorems.

Inspired by the aforementioned works, using the method of lower and upper solutions and the Krasnoselskii and Banach fixed point theorems, we study in this chapter, the existence and uniqueness of solutions of nonlinear fractional differential equation involving

Liouville - Caputo derivative:

$$\begin{cases} {}^{c}D^{\mu}(w(r) - h(r, w(r))) = f(r, w(r)), \quad 0 < r \le T, \\ w(0) = w_0 > h(0, w_0) > 0, \end{cases} \tag{2.1.1}$$

where $0 < \mu \le 1$ and $h, f : [0, T] \times [0, \infty) \to [0, \infty)$ are continuous functions.

2.2 Preliminaries

Let $B = C([0, T])$, be the Banach space of all real-valued continuous functions defined on the compact interval $[0, T]$, endowed with the norm $\|w\| = \max_{0 \le r \le T} |w(r)|$. Let K be a nonempty closed subset of B defined as $K = \{w \in B : \|w\| \le l, \; l > 0\}$. Let $p, q \in \mathbb{R}^+$ with $p < q$ and for any $w \in [p, q] \subset \mathbb{R}^+$, we define the upper and lower control functions respectively as follows

$$M(r, w) = \sup_{p \le \eta \le w} f(r, \eta), \quad m(r, w) = \inf_{w \le \eta \le q} f(r, \eta).$$

It is obvious that $m(r, w)$ and $M(r, w)$ are monotonic non-decreasing on $[p, q]$ and $m(r, w) \le f(r, w) \le M(r, w)$.

We give some definitions and their properties for our main results.

Definition 2.1. [30] The fractional integral of order $\mu > 0$ of a function $w : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$I^\mu w(r) = \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} w(s)\, ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2. [28, 53] The Liouville-Caputo fractional derivative of order $\mu > 0$ of a function $w : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$^cD^\mu w(r) = I^{n-\mu} w^n(r) = \frac{1}{\Gamma(n-\mu)} \int_0^r (r-s)^{n-\mu-1} w^{(n)}(s)\, ds,$$

where $n = [\mu] + 1$, provided right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3. *[30] Let $Re(\mu) > 0$, $w \in C^{n-1}([0, +\infty))$ and $w^{(n-1)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$(I^\mu\, {}^cD_0^\mu w)(r) = w(r) - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{k!} r^k.$$

In particular, when $0 < Re(\mu) < 1$, $(I^\mu\, {}^cD_0^\mu w)(r) = w(r) - w(0)$.

Lemma 2.4. *Let $w \in C([0, T])$, w' and $\frac{\partial h}{\partial r}$ exist, then $w(r)$ is a solution of (2.1.1) if and only if*

$$w(r) = w_0 - h(0, w_0) + h(r, w(r)) + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds. \tag{2.2.1}$$

Proof. Suppose $w(r)$ satisfies (2.1.1), then applying I^μ to both sides of (2.1.1), we have

$$I^\mu[{}^cD^\mu(w(r) - h(r, w(r)))] = I^\mu f(r, w(r)),\ 0 < r \le T.$$

In view of Lemma 2.3 and the initial condition of problem (2.1.1), we get

$$w(r) = w_0 - h(0, w_0) + h(r, w(r)) + \frac{1}{\Gamma(\mu)} \int_0^r (r - s)^{\mu-1} f(s, w(s))\, ds.$$

Conversely, suppose that $w(r)$ satisfies equation (2.2.1). Then applying ${}^cD^\mu$ to both sides of equation (2.2.1), we obtain

$$\begin{aligned} {}^cD^\mu w(r) &= {}^cD^\mu[w_0 - h(0, w_0) + h(r, w(r)) + \\ &\qquad \frac{1}{\Gamma(\mu)} \int_0^r (r - s)^{\mu-1} f(s, w(s))\, ds] \\ &= {}^cD^\mu h(r, w(r)) + {}^cD^\mu I^\mu f(r, w(r)) \\ &= {}^cD^\mu h(r, w(r)) + f(r, w(r)) \end{aligned}$$

$${}^cD^\mu[w(r) - h(r, w(r)] = f(r, w(r))..$$

Then ${}^cD^\mu[w(r) - h(r, w(r))] = f(r, w(r))$ and the initial condition $w(0) = w_0$ holds. □

2.3 Existence and uniqueness of solutions

In this section, first we need to construct two mappings, one is contraction and other is completely continuous. Now, define the operator $\Phi : K \to B$ by

$$\begin{aligned}(\Phi w)(r) &= w_0 - h(0, w_0) + h(r, w(r)) + \\ &\qquad \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds \\ &= (Pw)(r) + (Qw)(r),\end{aligned} \tag{2.3.1}$$

where the operator $P : K \to B$ is defined as,

$$(Pw)(r) = \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds,$$

and the operator $Q : K \to B$ is defined as,

$$(Qw)(r) = w(0) - h(0, w_0) + h(r, w(r)).$$

Throughout this chapter, we assume that the following holds:

(C1) $h, f \in C([0,T] \times [0,\infty), [0,\infty))$ and h is non-decreasing on w.

(C2) Let $w^*, w_* \in K$ such that $c \leq w_* \leq w^* \leq d$ and satisfying

$${}^cD^\mu(w^*(r) - h(r, w^*(r))) \geq M(r, w^*(r)),$$

$$^{c}D^{\mu}(w_*(r) - h(r, w_*(r))) \leq m(r, w_*(r)),$$

for any $r \in [0, T]$. The function w^* and w_* are respectively called a pair of upper and lower solutions for the equation (2.1.1).

(C3) For $u, v \in B$ and $r \in [0, T]$, there exist $\alpha \in (0, 1)$ and $\beta < 1$ such that

$$|h(r, u) - h(r, v)| \leq \alpha \|u - v\|,$$

$$|f(r, u) - f(r, v)| \leq \beta \|u - v\|.$$

We need the following lemmas to establish our results.

Lemma 2.5. *Assume that $[C1]$ holds. Then the operator $P : K \to B$ is completely continuous.*

Proof. By $[C1]$, f is continuous and nonnegative function, we get that $P : K \to B$ is continuous. The function $f : [0, T] \times K \to [0, \infty)$ is bounded, then $\exists\ \lambda > 0$ such that $0 \leq f(r, w(r)) \leq \lambda$. We obtain

$$\begin{aligned}
|(Pw)(r)| &\leq \frac{1}{\Gamma(\mu)} \int_0^r (r - s)^{\mu - 1} |f(s, w)|\, ds \\
&\leq \frac{\lambda}{\Gamma(\mu)} \left[\frac{(r - s)^{\mu}}{-\mu} \right]_0^r \\
&= \frac{\lambda r^{\mu}}{\Gamma(\mu + 1)} \leq \frac{\lambda T^{\mu}}{\Gamma(\mu + 1)}.
\end{aligned}$$

Hence $P(K)$ is uniformly bounded.

Now we will prove equicontinuity of P. Let $w \in K$, $\epsilon > 0$, $\delta > 0$

and for any $r_1, r_2 \in [0,T]$ with $r_1 < r_2$ such that $|r_2 - r_1| < \delta$. If $\delta = \left[\frac{\epsilon\Gamma(\mu+1)}{2\lambda}\right]^{\frac{1}{\mu}}$, then we have

$$
\begin{aligned}
|(Pw)(r_1) - (Pw)(r_2)| &\leq \frac{1}{\Gamma(\mu)} \int_0^{r_1} \left|(r_1 - s)^{\mu-1} - (r_2 - s)^{\mu-1}\right| |f(s,w)| \, ds \\
&\quad + \frac{1}{\Gamma(\mu)} \int_{r_1}^{r_2} \left|(r_2 - s)^{\mu-1}\right| |f(s,w(s)| \, ds \\
&\leq \frac{\lambda}{\Gamma(\mu)} \int_0^{r_1} [(r_1 - s)^{\mu-1} - (r_2 - s)^{\mu-1}] \, ds \\
&\quad + \frac{\lambda}{\Gamma(\mu)} \int_{r_1}^{r_2} (r_2 - s)^{\mu-1} \, ds \\
&= \frac{\lambda}{\Gamma(\mu+1)} [r_1^{\mu} - r_2^{\mu} + 2(r_2 - r_1)^{\mu}] \\
&\leq \frac{2\lambda}{\Gamma(\mu+1)} (r_2 - r_1)^{\mu} \\
&< \epsilon.
\end{aligned}
$$

Therefore $P(K)$ is equicontinuous. Then by Arzela-Ascoli theorem, $P : K \to B$ is completely continuous. □

Lemma 2.6. *Assume that $[C1]$ and $[C3]$ holds. Then the operator $Q : K \to B$ is contraction.*

Proof. By $[C1]$ and initial conditions of problem (2.1.1), the operator $Q : K \to B$ is continuous. For $u, v \in K$, we have

$$
|(Qu)(r) - (Qv)(r)| = |h(r,u(r)) - h(r,v(r))| \leq \alpha \|u - v\|.
$$

Thus $\|(Qu)(r) - (Qv)(r)\| \leq \alpha \|u - v\|$. Hence Q is contraction. □

Theorem 2.7. *Assume that* $[C1]$ *and* $[C2]$ *holds, then there exists at least one solution* $w(r) \in B$ *of the problem* (2.1.1) *satisfying* $w_*(r) \leq w(r) \leq w^*(r)$, $r \in [0,T]$.

Proof. Let $U = \{w \in K : w_*(r) \leq w(r) \leq w^*(r), r \in [0,T]\}$, endowed with the norm $\|w\| = \underset{0 \leq r \leq T}{max} |w(r)|$, then we have $\|w\| \leq l$. Hence U is a bounded, closed and convex subset of Banach space B. Moreover by $[C1]$, the continuity of h, f implies that the operator Φ defined by (2.3.1) is continuous on U. By Lemma 2.5, $P : U \to K$ is completely continuous. Also by Lemma 2.6, $Q : U \to K$ is contraction. Now, we show that if $u(r), v(r) \in U$ then $(Pu)(r) + (Qv)(r) \in U$. For any $u(r), v(r) \in U$, we have $w_*(r) \leq u(r), v(r) \leq w^*(r)$, then

$$\begin{aligned}(Pu)(r) + (Qv)(r) &= u_0 - h(0,u_0) + h(r, v(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, u(s))\, ds \\ &\leq u_0 - h(0,u_0) + h(r, w^*(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} M(s, w^*(s))\, ds \leq w^*(r),\end{aligned} \tag{2.3.2}$$

and

$$\begin{aligned}(Pu)(r) + (Qv)(r) &= u_0 - h(0,u_0) + h(r, v(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, u(s))\, ds\end{aligned}$$

$$\geq u_0 - h(0, u_0) + h(r, w_*(r)) + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} m(s, w_*(s))\, ds \geq w_*(r). \tag{2.3.3}$$

Thus from (2.3.2) and (2.3.3), $w_*(r) \leq (Pu)(r) + (Qv)(r) \leq w^*(r)$ implying that $(Pu)(r) + (Qv)(r) \in U$. Hence by Krasnoselskii fixed point theorem, there exists fixed point $w(r) \in U$ such that

$w(r) = (Pw)(r) + (Qw)(r)$, $r \in [0, T]$ in U.

Thus the problem (2.1.1) has at least one solution $w(r) \in U$ and $w_*(r) \leq w(r) \leq w^*(r)$, $r \in [0, T]$. □

Corollary 2.8. *Assume that, $[C1] - [C3]$ holds and there exists ζ_1, ζ_2, ζ_3, ζ_4 such that*

$$0 < \zeta_1 \leq h(r, w) \leq \zeta_2 < \infty, \quad (r, w(r)) \in [0, T] \times [0, \infty), \tag{2.3.4}$$

$$0 < \zeta_3 \leq f(r, w) \leq \zeta_4 < \infty, \quad (r, w(r)) \in [0, T] \times [0, \infty). \tag{2.3.5}$$

Then the problem (2.1.1) has at least one solution $w \in B$. Moreover,

$$w_0 - h(0, w_0) + \zeta_1 + \zeta_3 \frac{r^\mu}{\Gamma(\mu+1)} \leq w(r) \leq w_0 - h(0, w_0) + \zeta_2 + \zeta_4 \frac{r^\mu}{\Gamma(\mu+1)}. \tag{2.3.6}$$

Proof. By (2.3.5) and definition of control functions, we have

$$\zeta_3 \leq m[r,w] \leq M[r,w] \leq \zeta_4, (r,w(r)) \in [0,T] \times [0,\infty). \tag{2.3.7}$$

Now, we consider the equations

$$\begin{aligned} {}^cD_0^\mu[w(r)-\zeta_1] &= \zeta_3, \quad w(0)=w_0, \\ {}^cD_0^\mu[w(r)-\zeta_2] &= \zeta_4, \quad w(0)=w_0. \end{aligned} \tag{2.3.8}$$

Then, equation (2.3.8) are equivalent to

$$\begin{aligned} w(r) &= w_0 - h(0,w_0) + \zeta_1 + \frac{\zeta_3}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds \\ &= w_0 - h(0,w_0) + \zeta_1 + \zeta_3 \frac{r^\mu}{\Gamma(\mu+1)} \end{aligned}$$

and

$$\begin{aligned} w(r) &= w_0 - h(0,w_0) + \zeta_2 + \frac{\zeta_4}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds \\ &= w_0 - h(0,w_0) + \zeta_2 + \zeta_4 \frac{r^\mu}{\Gamma(\mu+1)}. \end{aligned}$$

Now taking into account (2.3.4), (2.3.7) we have

$$\begin{aligned} w_*(r) &= w_0 - h(0,w_0) + \zeta_1 + \frac{\zeta_3}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds \\ &\leq w_0 - h(0,w_0) + h(r,w_*) + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} m(r,w_*)\, ds \end{aligned}$$

and

$$w^*(r) = w_0 - h(0,w_0) + \zeta_2 + \frac{\zeta_4}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds$$

$$\geq w_0 - h(0, w_0) + h(r, w^*) + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} M(r, w^*)\, ds.$$

Then it is clear that, $w_*(r)$ and $w^*(r)$ are respectively the lower and upper solutions of equation (2.3.8). Therefore, an application of Theorem 2.7, yields that the problem (2.1.1) has at least one solution $w \in U \subset B$ and satisfies equation (2.3.6). □

Theorem 2.9. *Assume that* $[C1]$ *and* $[C3]$ *holds and*

$$\alpha + \frac{\beta T^\mu}{\Gamma(\mu+1)} < 1. \tag{2.3.9}$$

Then the problem (2.1.1) *has a unique solution* $w \in U$.

Proof. From Theorem 2.7 it follows that the problem (2.1.1) has at least one solution in U. Hence for uniqueness of solution, we need only to prove that the operator $\Phi : K \to B$ defined in equation (2.3.1) is a contraction on B. For all ξ_1, $\xi_2 \in U$, we have

$$\begin{aligned}
|\Phi(\xi_1) - \Phi(\xi_2)| &\leq |h(r, \xi_1(r)) - h(r, \xi_2(r))| \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} |f(s, \xi_1(s)) - f(s, \xi_2(s))|\, ds \\
&\leq \alpha\, |\xi_1 - \xi_2| + \beta\, |\xi_1 - \xi_2| \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds \\
&= \alpha\, |\xi_1 - \xi_2| + \beta\, |\xi_1 - \xi_2| \frac{r^\mu}{\Gamma(\mu+1)} \\
&\leq \left(\alpha + \frac{\beta T^\mu}{\Gamma(\mu+1)} \right) |\xi_1 - \xi_2|.
\end{aligned}$$

Thus,

$$\|\Phi(\xi_1) - \Phi(\xi_2)\| \le \left(\alpha + \frac{\beta T^\mu}{\Gamma(\mu+1)}\right)\|\xi_1 - \xi_2\|.$$

Hence by equation (2.3.9), the operator Φ is a contraction mapping. Then by contraction mapping principle, we conclude that the problem (2.1.1) has a unique solution $w \in U$. □

Example 2.1. *Consider the following nonlinear fractional differential equation*

$$\begin{cases} {}^cD^{\frac{1}{4}}\left[w(r) - \frac{1+w(r)}{3+w(r)}\right] = \frac{1}{2+r}\left[2 + \frac{rw(r)}{2+w(r)}\right], \quad 0 < r \le 1, \\ w(0) = 1, \end{cases} \tag{2.3.10}$$

where $w_0 = 1,\ T = 1,\ h(r,w) = \frac{1+w(r)}{3+w(r)},\ f(r,w) = \frac{1}{2+r}\left[2 + \frac{rw(r)}{2+w(r)}\right]$ *and* $h(0,w_0) = \frac{1}{2}$. *Since h is non-decreasing on w,* $\lim_{w\to\infty}\frac{1+w(r)}{3+w(r)} = 1$, $\lim_{w\to\infty}\frac{1}{2+r}\left[2 + \frac{rw(r)}{1+w(r)}\right] = 1$ *and* $\frac{1}{3} \le h(r,w) \le 1,\ \frac{2}{3} \le f(r,w) \le 1,$ *for* $(r,w) \in [0,1]\times[0,\infty)$. *Hence from Corollary 2.8, equation (2.3.10) has solution which satisfies* $w_*(r) \le w(r) \le w^*(r)$, *where* $w^*(r) = \frac{3}{2} + \frac{4r^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}$, $w_*(r) = \frac{5}{6} + \frac{8r^{\frac{1}{4}}}{3\Gamma(\frac{1}{4})}$ *are respectively the upper and lower solutions of* (2.3.10).

Also $\alpha + \frac{\beta T^{\frac{1}{4}}}{\Gamma(q)} \approx 0.3106 < 1$ *, then by Theorem 2.9 and (2.3.10) has a unique solution which is bounded by* $w_*(r)$ *and* $w^*(r)$.

2.4 Conclusion

Krasnoselskii and Banach fixed point theorems are applied to study the existence and uniqueness of solutions of nonlinear fractional differential equation involving Liouville - Caputo derivative. Lower and upper solutions for the considered problem are defined and applied to study existence and uniqueness of the problem (2.1.1).

Chapter 3

Conjugate-type Fractional Differential Equations with Integral Boundary Conditions

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3.1 Introduction

In the recent years, the theory of singular boundary value problems has become an important area of investigation [18, 65, 68, 69]. The existence of solutions by using various methods such as lower and upper solution method and fixed point theorem is proved. In [76] X. Zhang et al. obtained the existence and uniqueness of positive solutions when g has singularities at $r = 0$ and (or)1 by using monotone iterative method. In 2020 [60] S. Song et al. investigated the existence of extremal solutions by using monotone iterative technique coupled with lower and upper solutions for the problem

$$\begin{cases} -D^{\nu}_{0+}z(r) = g(r, z(r)), & r \in [0, 1], \\ z(0) = 0, \qquad z(1) = \int_0^1 z(s)\, d\eta(s), & \end{cases}$$

where $1 < \nu < 2$, D^{ν}_{0+} is the Riemann-Liouville fractional derivative and $\eta(r)$ is a positive measure function. Y. Wang et al. [66] studied the positive properties of the green function for the Dirichlet-type problem

$$\begin{cases} -D^{\nu}_{0+}z(r) + az(r) = g(r, z(r)), & 0 < r < 1, \\ z(0) = 0, \qquad z(1) = 0, & \end{cases}$$

where $1 < \nu < 2$, $a > 0$, D^{ν}_{0+} is the Riemann-Liouville fractional derivative. Y. Wang et al.[67] established the existence of positive solutions for resonant problem.

Inspired by the aforementioned works, in this chapter we prove the following problem have minimal and maximal solutions using monotone method:

$$\begin{cases} D^{\nu}_{0+}z(r) + g(r, z(r)) = 0, \quad 0 < r < 1, \quad l - 1 < \nu \leq l, \\ z^{(k)}(0) = 0, \quad 0 \leq k \leq l - 2, \quad z(1) = \displaystyle\int_0^1 z(s)\, d\eta(s), \end{cases} \tag{3.1.1}$$

where, D^{ν}_{0+} is the Riemann-Liouville fractional derivative of order $l \geq 2$, $l \in \mathbb{N}$, g has singularities at $r = 0$ and (or) 1, η is a function of bounded variation and $\int_0^1 z(s)d\eta(s)$ denotes the Riemann-Stieltjes integral of z with respect to η, $d\eta$ can be signed measure .

The layout of this chapter is as follows: In section 3.2, we present some basic definitions and lemmas that will be used to prove our main results. Section 3.3 is devoted to uniqueness of solution to problem (3.1.1) by using Banach contraction principle. In Section 3.4, we develop monotone iterative method and applied to obtain existence and uniqueness results for Riemann-Liouville fractional differential equations with integral boundary conditions.

3.2 Basic Results

In this section, we present some useful definitions and lemmas that will be used in the next section to attain existence and uniqueness results for the nonlinear problem (3.1.1).

Definition 3.1. A function $\dot{x}_0 \in C([0,\,1])$ is called a lower solution of the problem (3.1.1)

if it satisfies

$$\begin{cases} D^{\nu}_{0+}\dot{x}_0(r) + g(r, \dot{x}_0(r)) \geq 0, & 0 < r < 1, l-1 < \nu \leq l, \\ \dot{x}_0^{(k)}(0) = 0, \quad 0 \leq k \leq l-2, & \dot{x}_0(1) \leq \int_0^1 \dot{x}_0(s)\, d\eta(s). \end{cases} \tag{3.2.1}$$

Definition 3.2. A function $\dot{y}_0 \in C([0,\,1])$ is called a upper solution of the problem (3.1.1)

if it satisfies

$$\begin{cases} D^{\eta}_{0+}\dot{y}_0(r) + g(r, \dot{y}_0(r)) \leq 0, & 0 < r < 1, \quad l-1 < \nu \leq l, \\ \dot{y}_0^{(k)}(0) = 0, \quad 0 \leq k \leq l-2, & \dot{y}_0(1) \geq \int_0^1 \dot{y}_0(s)\, d\eta(s). \end{cases} \tag{3.2.2}$$

Denote

$$\rho(r) = \frac{\nu - 2}{\Gamma(\nu - 1)} + \sum_{k=1}^{\infty} \frac{r^k}{\Gamma((k+1)\nu - 2)}$$

It is easy to check that (see [66, 67])

$$\rho(0) = \frac{\nu - 2}{\Gamma(\nu - 1)} < 0$$

$$\rho^{'}(r) = \sum_{k=1}^{\infty} \frac{k r^{k-1}}{\Gamma((k+1)\nu - 2)} > 0, \text{ on } \quad (0, \infty)$$

and

$$\lim_{r \to +\infty} \rho(r) = +\infty$$

Therefore, there exist a unique $a^* > 0$ such that $\rho(a^*) = 0$.

Set

$$G_a(r) = r^{\nu-1} E_{\nu,\nu}(a r^{\nu}), \quad \text{where} \quad E_{\nu,\nu}(r) = \sum_{k=0}^{\infty} \frac{r^k}{\Gamma((k+1)\nu)} \tag{3.2.3}$$

is the Mittag-Leffler function ([30]).

For convenience, we list here the following assumptions.

$[B_1]$ the parameter a satisfies $a \in (0, a^*]$,

$[B_2]$ $\eta(r)$ is bounded variation in $(0, 1)$ such that $0 < \alpha \leq 1$,

$\alpha = \displaystyle\int_0^1 G_a(s)\, d\eta(s)$

and $0 \leq \zeta_\eta(s) = \displaystyle\int_0^1 H_a(r, s)\, d\eta(s)$, $0 < G_a(1) - \displaystyle\int_0^1 G_a(s)\, d\eta(s)$,

$[B_3]$ $g \in C((0, 1) \times [0, \infty), [0, \infty))$ and

$g(r, u) - g(r, v) \geq -a(u - v)$ for $\dot{x}_0 \leq u \leq v \leq \dot{y}_0$, $r \in (0, 1)$

Set

$$K_a(r,s) = H_a(r,s) + G_a(r)h^*(s)$$

where,

$$h^*(s) = \frac{\zeta_\eta(s)}{G_a(1) - \alpha}$$

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1), & \text{if } 0 \le s \le r \le 1 \\ G_a(r)G_a(1-s), & \text{if } 0 \le r \le s \le 1 \end{cases} \tag{3.2.4}$$

Lemma 3.3. *[66] Suppose that* $[B1]$ *holds and* $y \in L[0,1]$*. Then the problem*

$$\begin{cases} -D^{\nu}_{0+}z(r) + az(r) = q(r), & 0 < r < 1, \\ z(0) = 0, & z(1) = 0, \end{cases}$$

has a unique solution

$$z(r) = \int_0^1 H_a(r,s)q(s)\,ds,$$

where

$$H_a(r,s) = \frac{1}{G_a(1)} \begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1), & \text{if } 0 \le s \le r \le 1 \\ G_a(r)G_a(1-s), & \text{if } 0 \le r \le s \le 1 \end{cases}$$

Lemma 3.4. *Suppose that* $[B1]$, $[B2]$ *hold and* $y \in C([0,1])$. *Then linear fractional boundary value problem*

$$\begin{cases} D^{\nu}_{0+} z(r) - a z(r) + q(r) = 0, \quad 0 < r < 1, \quad l-1 < \nu \le l, \\ z^{(k)}(0) = 0, \quad 0 \le k \le l-2, \qquad z(1) = \displaystyle\int_0^1 z(s)\, d\eta(s), \end{cases} \tag{3.2.5}$$

has the following unique solution

$$z(r) = \int_0^1 K_a(r,s) q(s)\, ds.$$

Proof. First apply I^{ν} on linear equation (3.2.5) and using result, see in [30], we get

$$z(r) = -\int_0^r G_a(r-s) q(s)\, ds + C_0 G_a(r) + C_1 G_a^{'}(r) + C_2 G_a^{''}(r) + \ldots + C_{l-1} G_a^{(l-1)}(r) \tag{3.2.6}$$

Since $z(0) = 0$ then $C_{l-1} = 0$.

Similarly $z^{'}(0) = z^{''}(0) = \ldots = z^{l-2}(0) = 0$ gives

$$C_1 = C_2 = \ldots = C_{l-2} = 0$$

Then equation (3.2.6) becomes

$$z(r) = -\int_0^r G_a(r-s) q(s)\, ds + C_0 G(r).$$

Using $z(1) = \int_0^1 z(s)\, d\eta(s)$, we obtain

$$C_0 = \frac{1}{G_a(1)} \left[\int_0^1 z(s)\, d\eta(s) + \int_0^1 G_a(1-s) q(s)\, ds. \right]$$

Hence,

$$\begin{aligned}
z(r) = &- \int_0^r G_a(r-s) q(s)\, ds + \\
&\quad \frac{G_a(r)}{G_a(1)} \left[\int_0^1 z(s)\, d\eta(s) + \int_0^1 G_a(1-s) q(s)\, ds \right] \\
= &- \int_0^r G_a(r-s) q(s)\, ds + \frac{G_a(r)}{G_a(1)} \int_0^1 z(s)\, d\eta(s) + \\
&\frac{G_a(r)}{G_a(1)} \int_0^r G_a(1-s) q(s)\, ds + \frac{G_a(r)}{G_a(1)} \int_r^1 G_a(1-s) q(s)\, ds \\
= &\frac{1}{G_a(1)} \int_0^r \left[G_a(r) G_a(1-s) - G_a(1) G_a(r-s) \right] q(s)\, ds \\
&+ \frac{1}{G_a(1)} \int_r^1 G_a(r) G_a(1-s) q(s)\, ds + \frac{G_a(r)}{G_a(1)} \int_0^1 z(s)\, d\eta(s) \\
= &\int_0^1 H_a(r,s) q(s)\, ds + \frac{G_a(r)}{G_a(1)} \int_0^1 z(s)\, d\eta(s).
\end{aligned}$$

Let

$$\begin{aligned}
\int_0^1 z(s)\, d\eta(s) = &\int_0^1 [\int_0^1 H_a(s,\tau) q(\tau)\, d\tau] d\eta(s) + \\
&\int_0^1 \frac{G_a(s)}{G_a(1)} d\eta(s) \int_0^1 z(s)\, d\eta(s).
\end{aligned}$$

Therefore

$$[1 - \frac{1}{G_a(1)} \int_0^1 G_a(s)\, d\eta(s)] \int_0^1 z(s)\, d\eta(s) = \int_0^1 [\int_0^1 H_a q(\tau)\, d\tau] d\eta(s)$$

$$\int_0^1 z(s)\, d\eta(s) = \frac{1}{[1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\, d\eta(s)]} \int_0^1 \int_0^1 H_a q(\tau)\, d\tau d\eta(s)$$

Therefore

$$\begin{aligned}
z(r) = &\int_0^1 H_a(r,s) q(s)\, ds + \\
&\frac{G_a(r)}{G_a(1)} \left[\frac{1}{1 - \frac{1}{G_a(1)}\int_0^1 G_a(s)\, d\eta(s)} \right] \int_0^1 \int_0^1 H_a q(\tau)\, d\tau d\eta(s) \\
= &\int_0^1 H_a(r,s) q(s)\, ds + \\
&\frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\, d\eta(s)} \int_0^1 \int_0^1 H_a(s,\tau) q(\tau)\, d\tau d\eta(s) \\
= &\int_0^1 [H_a(r,s) + \frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\, d\eta(s)} \\
&\int_0^1 H_a(s,\tau) d\eta(\tau)] q(s)\, ds \\
z(r) = &\int_0^1 K_a(r,s) q(s)\, ds.
\end{aligned}$$

□

Lemma 3.5. *Suppose [B1], [B2] holds, then the function $K_a(r,s)$ has the following properties:*

1. $K_a(r,s) > 0 \quad \forall\ r, s \in (0,1)$

2. $\psi_2(s) r^{\nu-1} \leq K_a(r,s) \leq \psi_1(s) r^{\nu-1}, \quad \forall\ r, s \in (0,1)$

   where, $\psi_1(s) = G_a(1-s) + G_a(1) h^*(s), \quad \psi_2(s) = \frac{1}{\Gamma(\nu)} h^*(s).$

Proof. We need to prove that (2) holds. By equation (3.2.3)

$$\frac{r^{\nu-1}}{\Gamma(\nu)} \le G_a(r) = r^{\nu-1}\sum_{k=0}^{\infty}\frac{a^k r^{\nu k}}{\Gamma((k+1)\nu)} \le r^{\nu-1}G_a(1), \quad r \in (0,1) \tag{3.2.7}$$

$$G_a^{'}(r) = \sum_{k=0}^{\infty}\frac{a^k r^{(k+1)\nu-2}}{\Gamma((k+1)\nu-1)} > 0, \quad r \in (0,1) \tag{3.2.8}$$

$$\begin{aligned}
G_a^{''}(r) &= \sum_{k=0}^{\infty}\frac{a^k[(k+1)\nu-2]r^{(k+1)\nu-3}}{\Gamma((k+1)\nu-1)} \\
&= r^{\nu-3}\sum_{k=0}^{\infty}\frac{a^k[(k+1)\nu-2]r^{k\nu}}{\Gamma((k+1)\nu-1)} \\
&= r^{\nu-3}\left[\frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty}\frac{a^k r^{k\nu}[(k+1)\nu-2]}{\Gamma((k+1)\nu-1)}\right] \\
&= r^{\nu-3}\left[\frac{\nu-2}{\Gamma(\nu-1)} + \sum_{k=1}^{\infty}\frac{a^k r^{k\nu}}{\Gamma((k+1)\nu-2)}\right] \\
&= r^{\nu-3}\rho(ar^{\nu}) < r^{\nu-3}\rho(a) \le r^{\nu-3}\rho(a^*) = 0, \quad r \in (0,1)
\end{aligned} \tag{3.2.9}$$

which implies that $G_a(r)$ is strictly increasing on $(0,1)$ and $G_a^{'}(r)$ is strictly decreasing on $(0,1)$.

Therefore by (3.2.7) we have,

$$K_a(r,s) = H_a(r,s) + G_a(r)h^*(s)$$

$$\begin{aligned}
&\leq \frac{G_a(r)G_a(1-s)}{G_a(1)} + G_a(r)h^*(s) \\
&= \left[\frac{G_a(1-s)}{G_a(1)} + h^*(s)\right] G_a(r) \\
&\leq \left[\frac{G_a(1-s)}{G_a(1)} + h^*(s)\right] r^{\nu-1}G_a(1) \\
&= [G_a(1-s) + G_a(1)h^*(s)]r^{\nu-1} \\
&= \psi_1(s)r^{\nu-1} \qquad (3.2.10)
\end{aligned}$$

where, $\psi_1(s) = G_a(1-s) + G_a(1)h^*(s)$.

On the other hand, when $0 \leq r \leq s \leq 1$. Note that $G_a(0) = 0$ and monotonocity of $G_a(r)$, it is clear that

$$G_a(r)G_a(1-s) > 0 \qquad (3.2.11)$$

Hence $H_a(r,s) > 0$ and by $[B2]$, $K_a(r,s) > 0$ when $0 < r \leq s \leq 1$.

When $0 < s < r < 1$, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\left[G_a(r)G_a(1-s) - G_a(r-s)G_a(1)\right] &= -G_a(r)G_a^{'}(1-s) + \\
&\quad G_a(1)G_a^{'}(r-s) \\
&\geq [G_a(1) - G_a(r)]G_a^{'}(1-s)
\end{aligned} \qquad (3.2.12)$$

Integrating with respect to s, we obtain

$$G_a(r)G_a(1-s) - G_a(r-s)G_a(1) \geq \int_0^s [G_a(1) - G_a(r)]G_a^{'}(1-\mu)\, d\mu$$

$$= [G_a(1) - G_a(r)] \left[\frac{G_a(1-\mu)}{-1} \right]_0^s$$

$$= [G_a(1) - G_a(r)][G_a(1) - G_a(1-s)]$$

$$> 0 \qquad (3.2.13)$$

Then, by (3.2.4), (3.2.11), (3.2.13), we get

$$H_a(r,s) = G_a(r)G_a(1-s) - G_a(r-s)G_a(1) > 0 \quad r,s \in (0,1)$$

Now,

$$K_a(r,s) = H_a(r,s) + G_a(r)h^*(s) \geq G_a(r)h^*(s)$$
$$\geq \frac{r^{\nu-1}}{\Gamma(\nu)} h^*(s) = \psi_2(s) r^{\nu-1} > 0 \quad r,s \in (0,1)$$

where, $\psi_2(s) = \frac{1}{\Gamma(\nu)} h^*(s)$. Hence the proof. □

Lemma 3.6. *For* $0 < r_1 \leq r_2 < 1$,

1. $|G_a(r_2) - G_a(r_1)| < E_{\nu,\nu-1}(a)\,|r_2 - r_1| = G_a(1)|r_2 - r_1|$,
2. $|G_a(r_2 - s) - G_a(r_1 - s)| < E_{\nu,\nu-1}(a)\,|r_2 - r_1| = G_a(1)|r_2 - r_1|$,
3. $|H_a(r_2,s) - H_a(r_1,s)| < 2[G_a(1)]^2 |r_2 - r_1|$,
4. $|K_a(r_2,s) - K_a(r_1,s)| \leq \max_{0 \leq s \leq 1} |K_a(r_2,s) - K_a(r_1,s)|$
   $< G_a(1)[2G_a(1) + |h^*(s)|]|r_2 - r_1|$

Proof.

$$[1] \quad |G_a(r_2) - G_a(r_1)| = \left|r_2^{\nu-1}E_{\nu,\nu}(ar_2^{\nu}) - r_1^{\nu-1}E_{\nu,\nu}(ar_1^{\nu})\right|$$
$$= \left|r_2^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k r_2^{\nu k}}{\Gamma((k+1)\nu)} - r_1^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k r_2^{\nu k}}{\Gamma((k+1)\nu)}\right|$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu)}\left|r_2^{\nu(k+1)-1} - r_1^{\nu(k+1)-1}\right|$$

Applying mean value theorem, we get

$$r_2^{\nu(k+1)-1} - r_1^{\nu(k+1)-1} < [\nu(k+1)-1](r_2 - r_1).$$

Therefore

$$|G_a(r_2) - G_a(r_1)| < \sum_{k=o}^{\infty}\frac{a^k[\nu(k+1)-1]}{\Gamma((k+1)\nu)}|r_2 - r_1|$$
$$= \sum_{k=o}^{\infty}\frac{a^k}{\Gamma((k+1)\nu-1)}|r_2 - r_1|$$
$$= E_{\nu,\nu-1}(a)|r_2 - r_1|$$
$$= G_a(1)|r_2 - r_1|.$$

$$[2] \quad |G_a(r_2 - s) - G_a(r_1 - s)| = |(r_2 - s)^{\nu-1}E_{\nu,\nu}(a(r_2 - s)^{\nu}) -$$
$$(r_1 - s)^{\nu-1}E_{\nu,\nu}(a(r_1 - s)^{\nu})|$$
$$= |(r_2 - s)^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k(r_2 - s)^{\nu k}}{\Gamma((k+1)\nu)} -$$
$$(r_1 - s)^{\nu-1}\sum_{k=o}^{\infty}\frac{a^k(r_2 - s)^{\nu k}}{\Gamma((k+1)\nu)}|$$

$$= \sum_{k=o}^{\infty} \frac{a^k}{\Gamma((k+1)\nu)} |(r_2 - s)^{\nu(k+1)-1} -$$

$$(r_1 - s)^{\nu(k+1)-1}|.$$

Applying mean value theorem, we get

$$(r_2 - s)^{\nu(k+1)-1} - (r_1 - s)^{\nu(k+1)-1} < [\nu(k+1) - 1](r_2 - r_1).$$

Therefore

$$|G_a(r_2 - s) - G_a(r_1 - s)| < \sum_{k=o}^{\infty} \frac{a^k}{\Gamma((k+1)\nu - 1)} |r_2 - r_1|$$

$$= E_{\nu,\,\nu-1}(a)|r_2 - r_1| = G_a|r_2 - r_1|.$$

[3] $|H_a(r_2, s) - H_a(r_1, s)| = |[G_a(r_2)G_a(1-s) - G_a(1)G_a(r_2 - s)]$

$$- [G_a(r_1)G_a(1-s) - G_a(1)G_a(r_1 - s)]|$$

$$< G_a(1-s)|G_a(r_2) - G_a(r_1)| +$$

$$G_a(1)|G_a(r_1 - s) - G_a(r_2 - s)|$$

$$< G_a(1-s)E_{\nu,\,\nu}(a)|r_2 - r_1| +$$

$$G_a(1)E_{\nu,\,\nu}(a)|r_2 - r_1|$$

$$= E_{\nu,\,\nu}(a)[G_a(1-s) + G_a(1)]|r_2 - r_1|$$

$$= G_a(1)[G_a(1-s) + G_a(1)]|r_2 - r_1|$$

$$< 2[G_a(1)]^2|r_2 - r_1|.$$

$$
\begin{aligned}
[4]\ |K_a(r_2,s)-K_a(r_1,s)| &\le \max_{0\le s\le 1}|K_a(r_2,s)-K_a(r_1,s)| \\
&= \max_{0\le s\le 1}|[H_a(r_2,s)-H_a(r_1,s)]|+ \\
&\quad\ |[G_a(r_2)-G_a(r_1)]h^*(s)| \\
&< 2[G_a(1)]^2|r_2-r_1|+G_a(1)|r_2-r_1||h^*(s)| \\
&= G_a(1)[2G_a(1)+|h^*(s)|]|r_2-r_1|.
\end{aligned}
$$

Hence the proof. □

3.3 Main results

Let $\mathscr{C}=C([0,1])$ be endowed with the norm $\|z\|=\max\limits_{0\le r\le 1}|z(r)|$, then $(\mathscr{C},\|.\|)$ is a Banach space. Now, define the operator $T:\mathscr{C}\to\mathscr{C}$ by

$$(Tz)(r)=\int_0^1 K_a(r,s)g(s,z(s))\,ds.$$

Theorem 3.7. *A mapping $T:\mathscr{C}\to\mathscr{C}$ is uniformly continuous.*

Proof. The operator $T:\mathscr{C}\to\mathscr{C}$ is continuous in the view of non-negativeness and continuity of $K_a(r,s)$, $H_a(r,s)$ and $g(r,z)$. Let $S\subset\mathscr{C}$ be bounded i.e. $\exists$ a positive constants $M>0$ such that $\|z\|<M\ \forall\ z\in S$, Let $L^*=\max\limits_{0\le r\le 1}|g(r,z)|$ then by Lemma 3.5 the operator $T:S\to\mathscr{C}$ is bounded uniformly.

Now to prove T(S) is equicontinuous.

If $z \in S, \quad 0 < r_1 < r_2 < 1$ then

$$\begin{aligned}
|(Tz)(r_2) - (Tz)(r_1)| &= \left| \int_0^1 [K_a(r_2, s) - K_a(r_1, s)] g(s, z(s))\, ds \right| \\
&\leq \max_{0 \leq s \leq 1} \int_0^1 |K_a(r_2, s) - K_a(r_1, s)|\, |g(s, z(s))|\, ds \\
&< L^* G_a(1) |r_2 - r_1| \int_0^1 [2G_a(1) + |h^*(s)|]\, ds \\
&< L^* G_a(1) |r_2 - r_1| [2G_a(1) + \int_0^1 |h^*(s)|]\, ds.
\end{aligned}$$

Then $|(Tz)(r_2) - (Tz)(r_1)| \to 0$ uniformly as $r_1 \to r_2$. This shows that $T(S)$ is equicontinuous on $\mathscr{C}$. Then by Arzela-Ascoli theorem, the operator $T : \mathscr{C} \to \mathscr{C}$ is completely continuous. □

Theorem 3.8. *Assume that* $[B1]$, $[B2]$ *holds. If there exists non-negative constant* N^* *such that function* $g(r, x)$ *satisfies*

$$|g(r, x) - g(r, y)| \leq N^* |x - y|, \quad r \in (0, 1), \quad x, y \in \mathscr{C}$$

and let $\Lambda = \int_0^1 \psi(s)\, ds$ *then the nonlinear problem (3.1.1) has a unique fixed point.*

Proof. For any $x, y \in \mathscr{C}, \quad r, s \in (0, 1)$ and using Lemma 3.5

$$\begin{aligned}
\|Tx(r) - Ty(r)\| &= \max_{0 \leq r \leq 1} |Tx(r) - Ty(r)| \\
&= \max_{0 \leq r \leq 1} \left| \int_0^1 K_a(r, s)[g(r, x) - g(r, y)]\, ds \right| \\
&= \max_{0 \leq r \leq 1} \int_0^1 |K_a(r, s)| |g(r, x) - g(r, y)|\, ds
\end{aligned}$$

$$\leq r^{\nu-1} N^* \|x - y\| \int_0^1 \psi(s)\, ds$$
$$= r^{\nu-1} N^* \Lambda \|u - v\|$$

Then by Banach contraction mapping theorem, T has a unique fixed point in $\mathscr{C}$. Hence the nonlinear problem (3.1.1) has unique solution.

□

3.4 Monotone Iterative Method

In this section, we develop monotone iterative technique combined with the method of lower-upper solutions and we prove the existence and uniqueness theorem of solution for problem (3.1.1).

For $\dot{x}_0, \dot{y}_0 \in \mathscr{C}$ with $\dot{x}_0 \leq \dot{y}_0$ for $r \in (0, 1)$, we denote

$$\Omega^* = [\dot{x}_0, \dot{y}_0] = \{z \in \mathscr{C} : \dot{x}_0 \leq z(r) \leq \dot{y}_0, r \in (0, 1)\}$$

Lemma 3.9. *Assume that* $[B1]$, $[B2]$ *holds and* $z(r) \in \mathscr{C}$ *satisfies*

$$\begin{aligned} & -D^{\nu} z(r) + a z(r) \geq 0 \\ & z^{(k)}(0) = 0, \qquad z(1) \geq \int_0^1 z(s)\, d\eta(s) \end{aligned} \tag{3.4.1}$$

then for $r \in (0, 1)$, $z(r) \geq 0$.

Proof. Let $q(r) = -D^\nu z(r) + az(r)$ and $d = z(1) - \int_0^1 z(s)\,d\eta(s)$. Then from equation (3.4.1), we have $q(r) \geq 0$, $d \geq 0$. Then by Lemma 3.4, the problem (3.2.5) has unique solution which can be expressed as

$$z(r) = \int_0^1 K_a(r,s)q(s)\,ds$$
$$= \int_0^1 H_a(r,s)q(s)\,ds + \left[\frac{G_a(r)}{G_a(1) - \int_0^1 G_a(s)\,d\eta(s)}\right]\int_0^1 H_a(r,s)\,d\eta(s)$$

where,

$$H_a(r,s) = \frac{1}{G_a(1)}\begin{cases} G_a(r)G_a(1-s) - G_a(r-s)G_a(1), & \text{if } 0 \leq s \leq r \leq 1 \\ G_a(r)G_a(1-s), & \text{if } 0 \leq r \leq s \leq 1 \end{cases}$$

Then by Lemma 3.5, $H_a(r,s) \geq 0$ and $K_a(r,s) \geq 0 \ \forall\ r,s \in (0,1)$. Hence $z(r) \geq 0$, for all $r,s \in (0,1)$. □

Theorem 3.10. *Suppose* $[B1]$, $[B2]$, $[B3]$ *holds, then there exist monotone iterative sequences* $\{\dot{x}_m\}$, $\{\dot{y}_m\} \subset \Omega^*$ *such that* $\dot{x}_m \to \dot{x}$, $\dot{y}_m \to \dot{y}$ *as* $m \to \infty$ *uniformly in* Ω^* *and* $\dot{x}$, $\dot{y}$ *are minimal and maximal solutions of problem (3.1.1) in* Ω^* *respectively.*

Proof. For $\dot{x}_{m-1}$, $\dot{y}_{m-1} \in \mathscr{C}$, $m \geq 1$, we define two sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ respectively by the relations,

$$\begin{cases} D^{\nu}_{0+}\dot{x}_m(r) - a(\dot{x}_m(r) + \dot{x}_{m-1}(r)) + g(r, \dot{x}_{m-1}(r)) = 0, \quad 0 < r < 1 \\ \dot{x}^{(k)}_m(0) = 0, \qquad \dot{x}_m(1) = \displaystyle\int_0^1 \dot{x}_m(s)\, d\eta(s) \end{cases}$$

and

$$\begin{cases} D^{\nu}_{0+}\dot{y}_m(r) - a(\dot{y}_m(r) + \dot{y}_{m-1}(r)) + g(r, \dot{y}_{m-1}(r)) = 0, \quad 0 < r < 1, \\ \dot{y}^{(k)}_m(0) = 0, \qquad \dot{y}_m(1) = \displaystyle\int_0^1 \dot{y}_m(s)\, d\eta(s) \end{cases}$$

Then by Lemma 3.4, $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ are well defined. Firstly we need to show that $\dot{x}_0(r) \leq \dot{x}_1(r) \leq \dot{y}_1(r) \leq \dot{y}_0(r)$ for any $r \in (0, 1)$.

Set $\dot{p}(r) = \dot{x}_1(r) - \dot{x}_0(r)$ and by definition of $\dot{x}_1(r)$ with lower solution $\dot{x}_0(r)$ we get,

$$\begin{aligned} -D^{\nu}_{0+}\dot{p}(r) + a\dot{p}(r) &= -D^{\nu}_{0+}(\dot{x}_1(r) - \dot{x}_0(r)) + a(\dot{x}_1(r) + \dot{x}_0(r)) \\ &= -D^{\nu}_{0+}\dot{x}_1(r) + a(\dot{x}_1(r) + \dot{x}_0(r)) + D^{\nu}_{0+}\dot{x}_0(r) \\ &\geq -a(\dot{x}_1(r) + \dot{x}_0(r)) + g(r, \dot{x}_0(r)) + \\ &\quad a(\dot{x}_1(r) + \dot{x}_0(r)) - g(r, \dot{x}_0(r)) \\ &= 0 \\ Also, \dot{p}^{(k)}(0) &= \dot{\dot{x}}^{(k)}_1(0) - \dot{x}^{k}_0(0) = 0 \\ \dot{p}(1) &= \dot{x}_1(1) - \dot{x}_0(1) \end{aligned}$$

$$\geq \int_0^1 \dot{x}_1(s)\, d\eta(s) - \int_0^1 x_0(s)\, d\eta(s)$$
$$= \int_0^1 [\dot{x}_1(s) - \dot{x}_0(s)]\, d\eta(s) = \int_0^1 \dot{p}(s)\, d\eta(s)$$

Then by Lemma 3.9, $\dot{p}(r) \geq 0 \Rightarrow \dot{x}_1(r) \geq \dot{x}_0(r)$, $r \in (0,1)$.

Now to prove $\dot{y}_1(r) \leq \dot{y}_0(r)$ $\forall r \in (0,1)$, set $\dot{p}(r) = \dot{y}_1(r) - \dot{y}_0(r)$ and by definition of $\dot{y}_1(r)$ with upper solution $\dot{y}_0(r)$ we get,

$$-D^{\nu}_{0+}\dot{p}(r) + a\dot{p}(r) = -D^{\nu}_{0+}(\dot{y}_1(r) - \dot{y}_0(r)) + a(\dot{y}_1(r) + \dot{y}_0(r))$$
$$= -D^{\nu}_{0+}\dot{y}_1(r) + a(\dot{y}_1(r) + \dot{y}_0(r)) + D^{\nu}_{0+}\dot{y}_0(r)$$
$$\leq -a(\dot{y}_1(r) + \dot{y}_0(r)) + g(r, \dot{y}_0(r)) +$$
$$a(\dot{y}_1(r) + \dot{y}_0(r)) - g(r, \dot{y}_0(r))$$
$$= 0$$
$$Also, \dot{p}^{(k)}(0) = \dot{y}_1^{(k)}(0) - \dot{y}_0^{k}(0) = 0$$
$$\dot{p}(1) = \dot{y}_1(1) - \dot{y}_0(1)$$
$$\leq \int_0^1 \dot{y}_1(s)\, d\eta(s) - \int_0^1 \dot{y}_0(s)\, d\eta(s)$$
$$= \int_0^1 [\dot{y}_1(s) - \dot{y}_0(s)]\, d\eta(s) = \int_0^1 \dot{p}(s)\, d\eta(s)$$

Then by Lemma 3.9, $\dot{p}(r) \leq 0 \Rightarrow \dot{y}_1(r) \leq \dot{y}_0(r)$, $\forall r \in (0,1)$.

Now to prove $\dot{x}_1(r) \leq \dot{y}_1(r)$ $\forall r \in (0,1)$. Set $\dot{p}(r) = \dot{y}_1(r) - \dot{x}_1(r)$. Then by $[B3]$ and definition of $\dot{x}_1(r)$, $\dot{y}_1(r)$, we get

$$-D^{\nu}_{0+}\dot{p}(r) = -D^{\nu}_{0+}[\dot{y}_1(r) - \dot{x}_1(r)]$$

$$= -D^{\nu}_{0+}\dot{y}_1(r) - [-D^{\nu}_{0+}\dot{x}_1(r)]$$

$$= g(r, \dot{y}_0(r)) - a[\dot{y}_1(r) - \dot{y}_0(r)] -$$

$$[g(r, \dot{x}_0(r)) - a[\dot{x}_1(r) - \dot{x}_0(r)]]$$

$$= [g(r, \dot{y}_0(r)) - g(r, \dot{x}_0(r))] - a[\dot{y}_1(r) - \dot{y}_0(r)] +$$

$$a[\dot{x}_1(r) - \dot{x}_0(r)]$$

$$\geq -a(\dot{y}_0(r) - \dot{x}_0(r)) - a[\dot{y}_1(r) - \dot{y}_0(r)] +$$

$$a[\dot{x}_1(r) - \dot{x}_0(r)]$$

$$= -a(\dot{y}_1(r) - \dot{x}_1(r))$$

$$= -a\dot{p}(r)$$

$$Also, \dot{p}^{(k)}(0) = \dot{y}_1^{(k)}(0) - \dot{x}_1^{(k)}(0) = 0$$

$$\dot{p}(1) = \dot{y}_1(1) - \dot{x}_1(1) = \int_0^1 \dot{y}_1(s)\, d\eta(s) - \int_0^1 \dot{y}_1(s)\, d\eta(s)$$

$$= \int_0^1 \dot{p}(s)\, d\eta(s)$$

Then by Lemma 3.9, $\dot{p}(r) \geq 0 \Rightarrow \dot{x}_1(r) \leq \dot{y}_1(r)\ \forall r \in (0,1)$. Now by mathematical induction, it is easy to verify that

$$\dot{x}_0(r) \leq \dot{x}_1(r) \leq \dot{x}_2(r) \leq \ldots \leq \dot{x}_m(r) \leq \dot{y}_m(r) \leq \ldots \dot{y}_1(r) \leq \dot{y}_0(r).$$

Thus the sequences $\{\dot{x}_m\}$, $\{\dot{y}_m\}$ are uniformly bounded and monotonically non-decreasing and non-increasing in $\mathscr{C}$. Hence the point-wise limit exist and are given by $\underset{m\to\infty}{lim}\ \dot{x}_m(r) = \dot{x}(r)$, $\underset{m\to\infty}{lim}\ \dot{y}_m(r) = \dot{y}(r)$ on

$\mathscr{C}$. Next we claim that $\dot{x}(r)$ and $\dot{y}(r)$ are the minimal and maximal solutions of problem (3.1.1). Let $z(r)$ be any solution of problem (3.1.1) different from $\dot{x}(r)$ and $\dot{y}(r)$ in Ω^*. So there exists some i such that $\dot{x}_i(r) \leq z(r) \leq \dot{y}_i(r),\ r \in (0,1)$. Set $\dot{p}_1(r) = z(r) - \dot{x}_{i+1}(r)$. So that, by assumption $[B3]$, we obtain

$$\begin{aligned}
-D^\nu \dot{p}_1(r) &= -D^\nu z(r) - (-D^\nu \dot{x}_{i+1}) \\
&= g(r, z(r)) - [g(r, \dot{x}_i(r)) - a(\dot{x}_{i+1}(r) - \dot{x}_i(r))] \\
&= [g(r, z(r)) - g(r, \dot{x}_i(r))] + a(\dot{x}_{i+1}(r) - \dot{x}_i(r)) \\
&\geq -a(z(r) - \dot{x}_i(r)) + a(\dot{x}_{i+1}(r) - \dot{x}_i(r)) \\
&= -a[z(r) - \dot{x}_i(r) - \dot{x}_{i+1}(r) + \dot{x}_i(r)] \\
&= -a(z(r) - \dot{x}_{i+1}(r)) = -a\dot{p}_1(r) \\
\dot{p}_1^{(k)}(0) &= 0, \quad \dot{p}_1(1) = \int_0^1 \dot{p}_1(s)\, d\eta(s)
\end{aligned}$$

Then by Lemma 3.9, $\dot{p}_1(r) \geq 0$ implying that $\dot{x}_{i+1}(r) \leq z(r)$ for all i. Similarly set $\dot{p}_2(r) = \dot{y}_{i+1}(r) - z(r)$ and using $[B3]$ we obtain

$$\begin{aligned}
-D^\nu \dot{p}_2(r) &= -D^\nu \dot{y}_{i+1}(r) - (-D^\nu z(r)) \\
&= [g(r, \dot{y}_i(r)) - a(\dot{y}_{i+1}(r) - \dot{y}_i(r))] - g(r, z(r)) \\
&= [g(r, \dot{y}_i(r)) - g(r, z(r))] - a(\dot{y}_{i+1}(r) - \dot{y}_i(r)) \\
&\geq -a(\dot{y}_i(r) - z(r)) + a(\dot{y}_{i+1}(r) - \dot{y}_i(r)) \\
&= -a[\dot{y}_i(r) - z(r) - \dot{y}_{i+1}(r) + \dot{y}_i(r)]
\end{aligned}$$

$$= -a(\dot{y}_{i+1}(r) - z(r)) = -a\dot{p}_2(r)$$

$$\dot{p}_2^{(k)}(0) = 0, \quad \dot{p}_2(1) = \int_0^1 \dot{p}_2(s)\, d\eta(s)$$

Then by Lemma 3.9, $\dot{p}_2(r) \geq 0$ implying that $z(r) \leq \dot{y}_{i+1}(r)$ for all i. Hence $\dot{x}_{i+1}(r) \leq z(r) \leq \dot{y}_{i+1}(r)$, $r \in (0,1)$. Since $\dot{x}_0(r) \leq z(r) \leq \dot{y}_0(r)$ on $\mathscr{C}$. Hence by induction method, it follows that $\dot{x}_i(r) \leq z(r) \leq \dot{y}_i(r)$ for all i. Taking limit as $i \to \infty$, it follows that $\dot{x}(r) \leq z(r) \leq \dot{y}(r)$ on $\mathscr{C}$. Thus the functions $\dot{x}(r)$, $\dot{y}(r)$ are the extremal solutions of the problem (3.1.1). □

Next we prove uniqueness of solutions of the problem (3.1.1).

Theorem 3.11. *Assume that*

1. *$[B_1]$, $[B_2]$, $[B_3]$ holds,*

2. *there exists $a > 0$ such that the function g satisfies the condition*

$$g(r, v) - g(r, v^*) \leq a(v - v^*) \tag{3.4.2}$$

for $\dot{x}_0 \leq v \leq v^* \leq \dot{y}_0, r \in (0,1)$.

Then the problem (3.1.1) has a unique solution in Ω^*.

Proof. We know $\dot{x}(r) \leq \dot{y}(r)$ on $\mathscr{C}$. It is sufficient to prove that $\dot{x}(r) \geq \dot{y}(r)$. Consider $\dot{p}(r) = \dot{y}(r) - \dot{x}(r)$. Then we have

$$\begin{aligned} -D^{\nu}\dot{p}(r) &= -D^{\nu}\dot{y}(r) - (-D^{\nu}\dot{x}(r)) \\ &= g(r, \dot{y}(r)) - g(r, \dot{x}(r)) \\ &\leq -a(\dot{y}(r) - \dot{x}(r)) = -a\dot{p}(r) \end{aligned}$$

and

$$\dot{p}^{(k)}(0) = 0, \quad \dot{p}(1) = \int_0^1 \dot{p}(s)\, d\eta(s)$$

By Lemma 3.9, $\dot{p}(r) \leq 0$ implying that $\dot{y}(r) \leq \dot{x}(r)$.

Hence $\dot{x}(r) = \dot{y}(r)$ is the unique solution of problem (3.1.1). □

3.5 Conclusion

1. By implementing Banach contraction mapping theorem it is shown that the mapping T has a unique fixed point in $\mathscr{C}$.

2. Monotone iterative sequences $\{\dot{x}_m\}$ and $\{\dot{y}_m\}$ converging uniformly to $\dot{x}$ and $\dot{y}$ as $m \to \infty$ respectively.

3. Monotone technique developed is applied to prove that $\dot{x}$, $\dot{y}$ are minimal and maximal solutions of problem (3.1.1) in Ω^*.

4. Uniqueness of solutions of the nonlinear problem (3.1.1) with integral boundary conditions is also obtained.

Chapter 4

Nonlinear BVP Involving ψ-Caputo Derivative

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4.1 Introduction

In 2020, Derbazi et al. [20] developed monotone iterative technique to study the existence and uniqueness of solution for initial value problem of nonlinear fractional differential equations including ψ-Caputo derivative. Dhaigude et. al. [26] have proved the existence and uniqueness of solution of nonlinear boundary value problems for ψ-Caputo fractional differential equations by applying monotone iterative technique. Abdo et. al. [3] investigates the existence and uniqueness of solutions of boundary value problems for ψ-Caputo fractional differential equations.

Motivated by their works, in this chapter, we consider the existence and uniqueness of solutions of the following nonlinear boundary value problems [BVP] involving ψ-Caputo fractional derivative

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} z(r) &= f(r, z(r)), \quad r \in J = [a, b], \\ z(a) &= a^*, \quad z(b) = b^*, \end{aligned} \tag{4.1.1}$$

where ${}^cD_{a^+}^{\mu,\psi}$ is the $\psi-$Caputo fractional derivative of order $\mu, 0 < \mu \leq 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function and $a^*, b^* \in \mathbb{R}$. Monotone iterative technique combined with coupled lower-upper solutions is developed for nonlinear BVP (4.1.1) and qualitative properties of solutions such as existence-uniqueness are obtained.

This chapter is organized as follows: In section 4.2, some basic definitions and useful lemmas are given. In section 4.3, lower-upper solutions of boundary value problem for ψ-Caputo fractional differential equation are introduced. Monotone technique is developed and successfully applied to obtain existence-uniqueness of solution of nonlinear BVP (4.1.1).

4.2 Preliminaries

In this section, we deduce some preliminary results required in the next section to attain existence and uniqueness results for nonlinear BVP (4.1.1) involving ψ-Caputo fractional derivative. Let $J = [a, b]$, where $0 \leq a < b < \infty$, be a finite interval and $\psi : J \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(r) \neq 0$,for all $r \in J$.

Lemma 4.1. *[7] Let $\mu, \nu > 0$, and $z \in L^1(J, \mathbb{R})$. Then*

$$I^{\mu,\psi} I^{\nu,\psi} z(r) = I^{\mu+\nu,\psi} z(r) \; a.e., \; r \in J.$$

In particular, if $z \in C(J, \mathbb{R})$, then $I^{\mu,\psi} I^{\nu,\psi} z(r) = I^{\mu+\nu,\psi} z(r)$, $r \in J$.

Lemma 4.2. *[7] For $\mu > 0$, the following holds:*

If $z(r) \in C(J, \mathbb{R})$ then

$$^{c}D_{a^+}^{\mu,\psi} I^{\mu,\psi} z(r) = z(r),\ r \in J.$$

If $z \in C^{n-1}(J, \mathbb{R})$, $n-1 < \mu < n$, then

$$I_{a^+}^{\mu,\psi} {}^{c}D_{a^+}^{\mu,\psi} z(r) = z(r) - \sum_{k=0}^{n-1} \frac{z_{\psi}^{[k]}(a)}{k!} [\psi(r) - \psi(a)]^k, \quad r \in J.$$

Lemma 4.3. *[30] For $r > a$, $\mu \geq 0$, and $\nu > 0$, we have*

1. $I_{a^+}^{\mu,\psi} [\psi(r) - \psi(a)]^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu+\mu)} [\psi(r) - \psi(a)]^{\nu+\mu-1}$,
2. $^{c}D_{a^+}^{\mu,\psi} [\psi(r) - \psi(a)]^{\nu-1} = \frac{\Gamma(\nu)}{\Gamma(\nu-\mu)} [\psi(r) - \psi(a)]^{\nu-\mu-1}$
3. $^{c}D_{a^+}^{\mu,\psi} [\psi(r) - \psi(a)]^{k} = 0$, *for all* $k \in \{0, 1, \ldots, n-1\}$, $n \in \mathbb{N}$.

Lemma 4.4. *[3] If $\mu > 0$ and $z, \psi \in C[a, b]$, then*

1. $I_{a^+}^{\mu,\psi}(.)$ *is linear and bounded from* $C[a, b]$ *to* $C[a, b]$.
2. $I_{a^+}^{\mu,\psi} z(a) = \lim_{r \to a} I_{a^+}^{\mu,\psi} z(r) = 0$.

Lemma 4.5. *[3] Let $n-1 < \mu < n$, $g \in C(J, \mathbb{R})$ and ψ is increasing and $\psi'(r) \neq 0$, for all $r \in J$. A function $z(r) \in C^n[a, b]$ is a solution of the fractional boundary value problem*

$$^{c}D_{a^+}^{\mu,\psi} z(r) = g(r), \qquad r \in J,$$

$$z_{\psi}^{[k]}(a) = z_a^k, \quad k = 0, 1, 2, \ldots n-2; z_{\psi}^{[n-1]}(b) = z_b,$$

where z_a^k, $z_b \in \mathbb{R}$, if and only if $z(r)$ satisfies the following fractional integral equation

$$\begin{aligned} z(r) = \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r) - \psi(a)]^k + \\ \left[\frac{z_b}{(n-1)!} + \frac{g(a)[\psi(b) - \psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu - n + 2)}\right][\psi(r) - \psi(a)]^{n-1} \\ - \frac{[\psi(r) - \psi(a)]^{n-1}}{(n-1)!\Gamma(\mu - n + 1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n} g(s)\, ds \\ + \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1} g(s)\, ds. \end{aligned}$$

4.3 Main Results

In this section, we develop monotone iterative technique and prove the existence and uniqueness of solution of the nonlinear BVP (4.1.1) involving ψ-Caputo fractional derivative.

Lemma 4.6. *For a given $g \in C(J, R)$ and $\mu \in (n-1, n)$, with $n \in \mathbb{N}$. A function $z(r) \in C^{n-1}[a, b]$ is the solution of the linear fractional boundary value problem*

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} z(r) + mz(r) = g(r), \quad r \in J = [a, b], \\ z_{\psi}^{[k]}(a) = z_a^k, \quad k = 0, 1, 2, \ldots n-2; \\ z_{\psi}^{[n-1]}(b) = z_b, \end{aligned} \tag{4.3.1}$$

where $z_a^k, z_b \in \mathbb{R}$, if and only if $z(r)$ satisfies the following fractional integral equation

$$z(r) = \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r) - \psi(a)]^k + M[\psi(r) - \psi(a)]^{n-1} + \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1}[g(s) - mz(s)]\, ds \tag{4.3.2}$$

where,

$$M = \frac{z_b}{(n-1)!} + \frac{[g(a) - mz(a)][\psi(b) - \psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu - n + 2)} - \frac{1}{(n-1)!\Gamma(\mu - n + 1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds.$$

Proof. First assume that $z(r) \in C^{n-1}[a, b]$ be a solution to problem (4.3.1). By Lemma 4.2,we have

$$z(r) = c_0 + c_1[\psi(r) - \psi(a)] + c_2[\psi(r) - \psi(a)]^2 + \ldots + c_{n-1}[\psi(r) - \psi(a)]^{n-1} + \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1}[g(s) - mz(s)]\, ds. \tag{4.3.3}$$

Using (4.3.3) we get,

$$z_\psi^{[0]}(a) = z_a^0 = z_a = c_0.$$

Now

$$
\begin{aligned}
z'(r) &= c_1\psi'(r) + 2c_2[\psi(r)-\psi(a)]\psi'(r) + \ldots + \\
&\qquad (n-1)c_{n-1}[\psi(r)-\psi(a)]^{n-2}\psi'(r) \\
&\qquad\quad + \frac{d}{dr}\frac{1}{\Gamma(\mu)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}[g(s)-mz(s)]\,ds, \\
&= c_1\psi'(r) + 2c_2[\psi(r)-\psi(a)]\psi'(r) + \ldots + \\
&\qquad (n-1)c_{n-1}[\psi(r)-\psi(a)]^{n-2}\psi'(r) \\
&\qquad\quad - \frac{1}{\Gamma(\mu)}\psi'(a)\left[[\psi(r)-\psi(a)]^{\mu-1}[g(a)-mz(a)]\right] \\
&+ \frac{1}{\Gamma(\mu-1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-2}\psi'(r)[g(s)-mz(s)]\,ds. \\
\therefore z_\psi^{[1]}(r) &= \frac{z'(r)}{\psi'(r)} \\
&= c_1 + 2c_2[\psi(r)-\psi(a)] + \ldots + (n-1)c_{n-1}[\psi(r)-\psi(a)]^{n-2} \\
&\qquad\quad - \frac{1}{\Gamma(\mu)}\left[[\psi(r)-\psi(a)]^{\mu-1}[g(a)-mz(a)]\right] \\
&\qquad\quad + \frac{1}{\Gamma(\mu-1)}\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-2}[g(s)-mz(s)]\,ds. \\
\therefore z_\psi^{[1]}(a) &= z_a^1 = c_1 \Rightarrow c_1 = \frac{z_a^1}{1!}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
z_\psi^{[2]}(r) &= \frac{[z_\psi^{[1]}(r)]'}{\psi'(r)} \\
&= 2c_2 + 6c_3[\psi(r)-\psi(a)] + \ldots + \\
&\qquad (n-1)(n-2)c_{n-1}[\psi(r)-\psi(a)]^{n-3}
\end{aligned}
$$

$$- \frac{2}{\Gamma(\mu-1)} \left[[\psi(r) - \psi(a)]^{\mu-2}[g(a) - mz(a)] \right]$$
$$+ \frac{1}{\Gamma(\mu-2)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-3}[g(s) - mz(s)]\, ds.$$

$$\therefore z_\psi^{[2]}(a) = z_a^2 = 2c_2 \Rightarrow c_2 = \frac{z_a^2}{2!}$$

Repeating this process we get,

$$c_k = \frac{z_a^k}{k!}, \quad k = 0, 1, 2, \ldots, n-2.$$

Again,

$$z_\psi^{[n-1]}(r) = \frac{[z_\psi^{[n-2]}(r)]'}{\psi'(r)}$$
$$= (n-1)!c_{n-1} -$$
$$\frac{(n-1)}{\Gamma(\mu-n+2)} \left[[\psi(r) - \psi(a)]^{\mu-n+1}[g(a) - mz(a)] \right]$$
$$+ \frac{1}{\Gamma(\mu-n+1)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds$$

$$\therefore z_\psi^{[n-1]}(b) = z_b = (n-1)!c_{n-1} -$$
$$\frac{(n-1)}{\Gamma(\mu-n+2)} \left[[\psi(b) - \psi(a)]^{\mu-n+1}[g(a) - mz(a)] \right]$$
$$+ \frac{1}{\Gamma(\mu-n+1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds.$$

Therefore

$$c_{n-1} = \frac{z_b}{(n-1)!} + \frac{[g(a) - mz(a)][\psi(b) - \psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)}$$
$$- \frac{1}{(n-1)!\Gamma(\mu-n+1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds.$$

Hence equation (4.3.3) becomes

$$z(r) = \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r) - \psi(a)]^k + \frac{z_b}{(n-1)!} + \frac{[g(a) - mz(a)][\psi(b) - \psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu - n + 2)}[\psi(r) - \psi(a)]^{n-1} - \frac{[\psi(r) - \psi(a)]^{n-1}}{(n-1)!\Gamma(\mu - n + 1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds + \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1}[g(s) - mz(s)]\, ds.$$

This equation can also be written in the form

$$z(r) = \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r) - \psi(a)]^k + M[\psi(r) - \psi(a)]^{n-1} + \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1}[g(s) - mz(s)]\, ds,$$

where,

$$M = \frac{z_b}{(n-1)!} + \frac{[g(a) - mz(a)][\psi(b) - \psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu - n + 2)} - \frac{1}{(n-1)!\Gamma(\mu - n + 1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds.$$

To prove the converse, we apply ${}^cD_{a^+}^{\mu,\psi}$ to both sides of equation (4.3.2) and using Lemma 4.2, we obtain

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} z(r) &= {}^cD_{a^+}^{\mu,\psi} I_{a^+}^{\mu,\psi}[g(r) - mz(r)] \\ &= g(r) - mz(r) \end{aligned}$$

$$\Rightarrow {}^{c}D_{a^+}^{\mu,\psi} z(r) + mz(r) = g(r).$$

It is clear that $z_a^0 = z_a$. Also the direct computations leads to

$$\begin{aligned} z_\psi^{[1]}(r) &= \frac{z'(r)}{\psi'(r)} \\ &= \sum_{k=1}^{n-2} \frac{z_a^k}{(k-1)!}[\psi(r)-\psi(a)]^{k-1} + (n-1)M[\psi(r)-\psi(a)]^{n-2} \\ &\qquad + \frac{1}{\Gamma(\mu-1)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-2}[g(s)-mz(s)]\,ds, \end{aligned}$$

and so $z_\psi^{[1]}(a) = z_a^1$.

$$\begin{aligned} z_\psi^{[2]}(r) &= \frac{[z_\psi^{[1]}(r)]'}{\psi'(r)} \\ &= \sum_{k=2}^{n-2} \frac{z_a^k}{(k-2)!}[\psi(r)-\psi(a)]^{k-2} + \\ &\quad (n-1)(n-2)M[\psi(r)-\psi(a)]^{n-3} \\ &\qquad + \frac{1}{\Gamma(\mu-2)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-3}[g(s)-mz(s)]\,ds, \end{aligned}$$

and so $z_\psi^{[2]}(a) = z_a^2$. Repeating this process, we write

$$\begin{aligned} z_\psi^{[n-2]}(r) &= \frac{(z^{[n-3]})'(r)}{\psi'(r)} \\ &= z_a^{n-2} + (n-1)!M[\psi(r)-\psi(a)] \\ &+ \frac{1}{\Gamma(\mu-n+2)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-n+1}[g(s)-mz(s)]\,ds. \end{aligned} \tag{4.3.4}$$

Taking $r \to a$ in equation (4.3.4), from continuity of $g(r)$ and using Lemma 4.4, we conclude that $z_\psi^{[n-2]}(a) = z_a^{n-2}$. Now

$$\begin{aligned} z_\psi^{[n-1]}(r) &= \frac{(z^{[n-2]})'(r)}{\psi'(r)} \\ &= (n-1)![\frac{z_b}{(n-1)!} + \frac{[g(a) - mz(a)][\psi(b) - \psi(a)]^{\mu-n+1}}{(n-2)!\Gamma(\mu-n+2)} \\ &- \frac{1}{(n-1)!\Gamma(\mu-n+1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds] \\ &+ \frac{1}{\Gamma(\mu-n+1)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds \\ &= z_b - \frac{1}{\Gamma(\mu-n+1)} \int_a^b \psi'(s)[\psi(b) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds \\ &+ \frac{1}{\Gamma(\mu-n+1)} \int_a^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-n}[g(s) - mz(s)]\, ds. \end{aligned} \tag{4.3.5}$$

Taking $r \to b$ in equation (4.3.5), from continuity of g and using Lemma 4.4, we conclude that $z_\psi^{[n-1]}(b) = z_b$. □

Lemma 4.7. *For a given $g(r) \in C(J, \mathbb{R})$ and $\mu \in (n-1, n]$, with $n \in \mathbb{N}$, the linear boundary value problem (4.3.1) has unique solution (4.3.2). Moreover, the explicit solution of the Volterra integral equation (4.3.2) can be represented by*

$$\begin{aligned} z(r) = &\sum_{k=0}^{n-2} z_a^k [\psi(r) - \psi(a)]^k E_{\mu,k+1}(-m(\psi(r) - \psi(a))^\mu) \\ &+ M\Gamma(n)[\psi(r) - \psi(a)]^{n-1} E_{\mu,n}(-m(\psi(r) - \psi(a))^\mu) \end{aligned}$$

$$+\int_0^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}E_{\mu,\mu}(-m(\psi(r)-\psi(a))^{\mu})g(s)\,ds, \tag{4.3.6}$$

where $E_{\mu,\nu}(.)$ is the two-parameter Mittag-Leffer function.

Proof. By Lemma 4.6, linear BVP (4.3.1) has a solution (4.3.2).

Note that the equation (4.3.2) can be written in the following form

$$z(r) = T[z(r)],$$

where the operator T is defined by

$$T[z(r)] = \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} - mI_{a^+}^{\mu,\psi}z(r)+$$

$$I_{a^+}^{\mu,\psi}g(r).$$

Let $n \in \mathbb{N}$ and $x, y \in C(J,\mathbb{R})$. We have

$$\begin{aligned}
\left|T^i(x)(r) - T^i(y)(r)\right| &= \left|-mI_{a^+}^{\mu,\psi}\left(T^{i-1}(x)(r) - T^{i-1}(y)(r)\right)\right| \\
&= \left|-mI_{a^+}^{\mu,\psi}\left(-mI_{a^+}^{\mu,\psi}\left(T^{i-2}(x)(r) - T^{i-2}(y)(r)\right)\right)\right| \\
&\vdots \\
&= \left|(-m)^i I_{a^+}^{i\mu,\psi}((x)(r) - (y)(r))\right| \\
&\leq \frac{(m[\psi(b)-\psi(a)]^{\mu})^i}{\Gamma(i\mu+1)}\left\|x-y\right\| \\
&= \frac{(m^i[\psi(b)-\psi(a)]^{i\mu})}{\Gamma(i\mu+1)}\left\|x-y\right\|,
\end{aligned}$$

for every $i \in N$ and $x, y \in A$. Let $\theta_i = \frac{(m^i[\psi(b)-\psi(a)]^{i\mu})}{\Gamma(i\mu+1)}$. Using generalized Mittag-Leffler functions, we have

$$\sum_{i=0}^{\infty} \theta_i = E_\mu(m(\psi(b)-\psi(a))^\mu).$$

Hence series $\sum_{i=0}^{\infty} \theta_i$ converges. Thus the mapping T^i is a contraction. Applying Weissinger's fixed point theorem, it follows that T has a unique fixed point. Hence equation (4.3.1) has unique solution $z(r)$. Applying method of successive approximations to prove that the integral equation (4.3.2) can be expressed by equation (4.3.6). For this, set

$$\begin{aligned}
z_0(r) &= \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1}, \\
z_m(r) &= z_0(r) - \frac{r}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} z_{m-1}(s)\, ds \\
&\quad + \frac{1}{\Gamma(\mu)} \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} g(s)\, ds.
\end{aligned}$$

$$\begin{aligned}
\therefore z_1(r) &= z_0(r) - mI_{a^+}^{\mu,\psi} z_0(r) + I_{a^+}^{\mu,\psi} g(r) \\
&= \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} - \\
&\quad m\sum_{k=0}^{n-2} \frac{z_a^k}{k!} \frac{\Gamma(k+1)}{\Gamma(\mu+k+1)}[\psi(r)-\psi(a)]^{\mu+k} \\
&\quad - mM\frac{\Gamma(n)}{\Gamma(\mu+n)}[\psi(r)-\psi(a)]^{\mu+n-1} + I_{a^+}^{\mu,\psi} g(r) \\
&= \sum_{k=0}^{n-2} \frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k + M[\psi(r)-\psi(a)]^{n-1} -
\end{aligned}$$

$$m\sum_{k=0}^{n-2}\frac{z_a^k}{\Gamma(\mu+k+1)}[\psi(t)-\psi(a)]^{\mu+k}$$
$$-mM\frac{\Gamma(n)}{\Gamma(\mu+n)}[\psi(r)-\psi(a)]^{\mu+n-1}+I_{a^+}^{\mu,\psi}g(r).$$

$$\therefore z_2(r)=z_0(t)-mI_{a^+}^{\mu,\psi}z_1(r)+I_{a^+}^{\mu,\psi}g(r)$$
$$=\sum_{k=0}^{n-2}\frac{z_a^k}{k!}[\psi(r)-\psi(a)]^k+M[\psi(r)-\psi(a)]^{n-1}-$$
$$m\sum_{k=0}^{n-2}\frac{z_a^k}{\Gamma(\mu+k+1)}[\psi(r)-\psi(a)]^{\mu+k}$$
$$-mM\frac{\Gamma(n)}{\Gamma(\mu+n)}[\psi(r)-\psi(a)]^{\mu+n-1}$$
$$+m^2\sum_{k=0}^{n-2}\frac{z_a^k\Gamma(\mu+k+1)}{\Gamma(\mu+k+1)\Gamma(2\mu+k+1)}[\psi(r)-\psi(a)]^{2\mu+k}$$
$$+m^2M\frac{\Gamma(n)\Gamma(\mu+n)}{\Gamma(\mu+n)\Gamma(2\mu+n)}[\psi(r)-\psi(a)]^{2\mu+k-1}-$$
$$mI_{a^+}^{2\mu,\psi}g(r)+I_{a^+}^{\mu,\psi}g(r)$$
$$=\sum_{l=0}^{2}\sum_{k=0}^{n-2}\frac{(-m)^l z_a^k}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu+k}+$$
$$\sum_{l=0}^{2}\frac{(-m)^l M\Gamma(n)}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu+n-1}$$
$$+\int_a^r\psi'(s)\sum_{l=0}^{2}\frac{(-m)^l[\psi(r)-\psi(s)]^{l\mu+\mu-1}}{\Gamma(l\mu+n)}g(s)\,ds.$$

Continuing this process, we derive the following relation

$$z_m(r)=\sum_{l=0}^{q}\sum_{k=0}^{n-2}\frac{(-m)^l z_a^k}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu+k}+$$
$$\sum_{l=0}^{q}\frac{(-m)^l M\Gamma(n)}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu+n-1}$$

$$+\int_a^r \psi'(s)\sum_{l=0}^{q-1}\frac{(-m)^l[\psi(r)-\psi(s)]^{l\mu+\mu-1}}{\Gamma(l\mu+n)}g(s)\,ds.$$

Taking limit as $q\to\infty$, we obtain the following explicit solution $z(r)$ to the integral equation (4.3.6).

$$\begin{aligned}
z(r) &= \sum_{l=0}^{\infty}\sum_{k=0}^{n-2}\frac{(-m)^l z_a^k}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu+k}+\\
&\quad \sum_{l=0}^{\infty}\frac{(-m)^l M\Gamma(n)}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu+n-1}\\
&\quad +\int_a^r \psi'(s)\sum_{l=0}^{\infty}\frac{(-m)^l[\psi(r)-\psi(s)]^{l\mu+\mu-1}}{\Gamma(l\mu+n)}g(s)\,ds\\
&= \sum_{k=0}^{n-2} z_a^k[\psi(r)-\psi(a)]^k\sum_{l=0}^{\infty}\frac{(-m)^l}{\Gamma(l\mu+k+1)}[\psi(r)-\psi(a)]^{l\mu}\\
&\quad + M\Gamma(n)[\psi(r)-\psi(a)]^{n-1}\sum_{l=0}^{\infty}\frac{(-m)^l}{\Gamma(l\mu+n)}[\psi(r)-\psi(a)]^{l\mu}\\
&\quad +\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}\sum_{l=0}^{\infty}\frac{(-m)^l}{\Gamma(l\mu+\mu)}[\psi(r)-\psi(s)]^{l\mu}g(s)\,ds.\\
\therefore z(r) &= \sum_{k=0}^{n-2} z_a^k[\psi(r)-\psi(a)]^k E_{\mu,k+1}(-m(\psi(r)-\psi(a))^{\mu})\\
&\quad + M\Gamma(n)[\psi(r)-\psi(a)]^{n-1}E_{\mu,n}(-m(\psi(r)-\psi(a))^{\mu})\\
&\quad +\int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}E_{\mu,\mu}(-m(\psi(r)-\psi(s))^{\mu})g(s)\,ds.
\end{aligned}$$

This proves the Lemma. □

Lemma 4.8. *(Comparison Result). Let $\mu \in (0,1]$ and $m \in \mathbb{R}$. If $p \in C(J,\mathbb{R})$ satisfies the following inequalities*

$$^{c}D_{a^+}^{\mu,\psi}p(r) \geq -mp(r), p(a) \geq 0, \quad p(b) \geq 0, \quad r \in J, \tag{4.3.7}$$

then $p(r) \geq 0$ for all $r \in J$.

Proof. Let $g(r) = {}^{c}D_{a^+}^{\mu,\psi}p(r) + mp(r)$ and $p(a) = a^*$, $p(b) = b^*$, where $a^*, b^* \in \mathbb{R}$. Then from Equation (4.3.7), $g(r) \geq 0$ and $a^* \geq 0$, $b^* \geq 0$. We know that $E_{\mu,1}(z) \geq 0$, $E_{\mu,\mu}(z) \geq 0$ for all $\mu \in (0,1]$, $z \in \mathbb{R}$ (see [53]) and $M \geq 0$. Then using the integral representation (4.3.6), we obtain that $p(r) \geq 0$ for all $r \in J$. □

Definition 4.9. A function $x_0 \in C(J,\mathbb{R})$ is said to be a lower solution of the nonlinear BVP (4.1.1), if it satisfies

$$^{c}D_{a^+}^{\mu,\psi}x_0(r) \leq f(r,x_0), \quad x_0(a) \leq a^*, \quad x_0(b) \leq b^*, \quad r \in J,$$

Definition 4.10. A function $y_0 \in C(J,\mathbb{R})$ is said to be a upper solution of the nonlinear BVP (4.1.1), if it satisfies

$$^{c}D_{a^+}^{\mu,\psi}y_0(r) \geq f(r,y_0), \quad y_0(a) \geq a^*, \quad y_0(b) \geq b^*, \quad r \in J,$$

Theorem 4.11. *Let $f(r,x(r)) \in C(J \times \mathbb{R},\mathbb{R})$. Assume that,*

(H_1) *There exist $x_0, y_0 \in C(J,\mathbb{R})$ such that x_0 and y_0 are lower and* *upper solutions of nonlinear BVP (4.1.1), respectively, with* $x_0 \leq y_0$, $r \in J$.

(H_2) *There exist a constant* $m \in \mathbb{R}$ *such that*

$$f(r,y) - f(r,x) \geq -m(y-x) \quad \textit{for} \quad x_0 \leq x \leq y \leq y_0.$$

Then there exist monotone iterative sequences $\{x_n\}$ *and* $\{y_n\}$ *converging uniformly on the interval* J *to the extremal solutions of nonlinear BVP (4.1.1) in the sector* $[x_0, y_0]$, *where*

$$[x_0, y_0] = \{z \in C(J,\mathbb{R}); x_0(r) \leq z(r) \leq y_0(r),\, r \in J\}.$$

Proof. For any $\omega \in [x_0, y_0]$, we consider the following linear BVP of fractional order

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} z(r) &= f(r,\omega(r)) - m(z(r) - \omega(r)), \quad r \in J, \\ z(a) &= a^*, \quad z(b) = b^*, \end{aligned} \tag{4.3.8}$$

Then the linear BVP (4.3.8) has unique solution $z(r)$.

Define the iterates as follows and construct the sequences $\{x_n\}$ and $\{y_n\}$:

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} x_{n+1}(r) &= f(r,x_n) - m(x_{n+1}(r) - x_n(r)), \quad r \in J, \\ x_{n+1}(a) &= a^*, \quad x_{n+1}(b) = b^*, \end{aligned} \tag{4.3.9}$$

and

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} y_{n+1}(t) &= f(r, y_n) - m(y_{n+1}(r) - y_n(r)), \quad r \in J, \\ y_{n+1}(a) &= a^*, \quad y_{n+1}(b) = b^*. \end{aligned} \tag{4.3.10}$$

Firstly we need to show that $x_0(r) \leq x_1(r) \leq y_1(r) \leq y_0(r)$ for any $r \in J$.

Set $p(r) = x_1(r) - x_0(r)$ and from (4.3.10) with lower solution x_0, we obtain

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} p(r) &= {}^cD_{a^+}^{\mu,\psi} x_1(r) - {}^cD_{a^+}^{\mu,\psi} x_0(r) \\ &\geq f(r, x_0(r)) - m(x_1(r) - x_0(r)) - f(r, x_0(r)) \\ &= -m(x_1(r) - x_0(r)) \\ &= -mp(r), \\ \text{and}\, p(a) &= x_1(a) - x_0(a) = a^* - x_0(a) \geq 0, \\ p(b) &= x_1(b) - x_0(b) = b^* - x_0(b) \geq 0. \end{aligned}$$

Then by Lemma 4.8, $p(r) \geq 0$, for $r \in J$, implies that $x_0(r) \leq x_1(r)$.

Similarly, set $p(r) = y_0(r) - y_1(r)$. From (4.3.9) and definition of upper solution, we obtain

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} p(r) &= {}^cD_{a^+}^{\mu,\psi} y_0(r) - {}^cD_{a^+}^{\mu,\psi} y_1(r) \\ &\geq f(r, y_0(r)) - f(r, y_0(r)) + m(y_1(r) - y_0(r)) \end{aligned}$$

$$= -m(y_0(r) - y_1(r))$$

$$= -mp(r),$$

$$\text{and}\, p(a) \geq 0, p(b) \geq 0.$$

Then by Lemma 4.8, $p(r) \geq 0$, for $r \in J$, implies that $y_1(r) \leq y_0(r)$ for $r \in J$.

Now to prove, $x_1(r) \leq y_1(r)$ for $r \in J$, set $p(r) = y_1(r) - x_1(r)$. From (4.3.9), (4.3.10) and (H_2), we get

$$\begin{aligned}
{}^cD_{a^+}^{\mu,\psi} p(r) &= {}^cD_{a^+}^{\mu,\psi} y_1(r) - {}^cD_{a^+}^{\mu,\psi} x_1(r) \\
&= f(r, y_0(r)) - f(r, x_0(r)) - m(y_1(r) - y_0(r)) + \\
&\qquad m(x_1(r) - x_0(r)) \\
&\geq -m(y_0(r) - x_0(r)) - m(y_1(r) - y_0(r)) + \\
&\qquad m(x_1(r) - x_0(r)) \\
&= -mp(r),
\end{aligned}$$

$$\text{and}\, p(a) = 0,\ p(b) = 0.$$

Then by Lemma 4.8, $p(r) \geq 0$, for $r \in J$, implies that $x_1(r) \leq y_1(r)$ for $r \in J$. Thus $x_0(r) \leq x_1(r) \leq y_1(r) \leq y_0(r)$ for any $r \in J$. Assume that $n > 1$, $x_{n-1}(r) \leq x_n(r) \leq y_n(r) \leq y_{n-1}(r)$ for any $r \in J$. We claim that $x_n(r) \leq x_{n+1}(r) \leq y_{n+1}(r) \leq y_n(r)$ for any $r \in J$. To prove this, set $p(r) = x_{n+1}(r) - x_n(r)$.

$$\begin{aligned}
{}^cD^{\mu,\psi}_{a^+}p(r) &= {}^cD^{\mu,\psi}_{a^+}x_{n+1}(r) - {}^cD^{\mu,\psi}_{a^+}x_n(r) \\
&= f(r, x_n(r)) - m(x_{n+1}(r) - x_n(r)) - f(r, x_{n-1}(r)) + \\
&\qquad m(x_n(r) - x_{n-1}(r)) \\
&\geq -m(x_n(r) - x_{n-1}(r)) - m(x_{n+1}(r) - x_n(r)) - \\
&\qquad m(x_n(r) - x_{n-1}(r)) \\
&= -m(x_{n+1}(r) - x_n(r)) \\
&= -mp(r), \\
\text{and}\, p(a) &= 0,\, p(b) = 0.
\end{aligned}$$

Then by Lemma 4.8, $p(r) \geq 0$, for $r \in J$, implies that $x_n(r) \leq x_{n+1}(r)$. Similarly we prove $x_{n+1}(r) \leq y_{n+1}(r)$ and $y_{n+1}(r) \leq y_n(r)$. By principle of mathematical induction, we have

$$x_0(r) \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n \leq \ldots \leq y_n \leq y_{n-1} \leq \ldots \leq y_1 \leq y_0(r).$$

Thus the sequences $\{x_n\}$ is monotone nondecreasing and bounded above by $y_0(r)$ and the sequences $\{y_n\}$ is monotone nonincreasing and bounded below by $x_0(r)$. Therefore the pointwise limit exist and these limits are denoted by x_*, y_*. Since sequences $\{x_n\}$, $\{y_n\}$ are of continuous functions defined on the compact set $[a, b]$. Hence by Dini's theorem, the convergence is uniform. That is

$$\lim_{n\to\infty} x_n(r) = x_*(r) \quad \text{and} \quad \lim_{n\to\infty} y_n(r) = y_*(r),$$

uniformly on $r \in J$ and the limit functions x_*, y_* satisfy BVP (4.1.1). Using corresponding fractional Volterra integral equations

$$\begin{aligned} x_{n+1}(r) = a^* E_{\mu,1}(-m(\psi(r)-\psi(a))^\mu) + b^* E_{\mu,1}(-m(\psi(r)-\psi(a))^\mu) \\ + \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^\mu) \\ [f(s, x_n(s) + m x_n(s)]ds, \quad r \in J, \\ y_{n+1}(r) = a^* E_{\mu,\mu}(-m(\psi(r)-\psi(a))^\mu) + b^* E_{\mu,\mu}(-m(\psi(r)-\psi(a))^\mu) \\ + \int_a^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^\mu) \\ [f(s, y_n(s) + m y_n(s)]ds, \quad r \in J, \end{aligned}$$

it follows that x_*, y_* are solutions of (4.3.9) and (4.3.10) respectively. Next prove that x_* and y_* are minimal and maximal solutions of BVP (4.1.1) in the sector $[x_0, y_0]$. Let $w \in [x_0, y_0]$ be any solution of BVP (4.1.1). Assume that for some $n \in N$, $x_n(r) \le w(r) \le y_n(r)$, $r \in J$. Set $p(r) = w(r) - x_{n+1}(r)$. We have

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} p(r) &= {}^cD_{a^+}^{\mu,\psi} w(r) - {}^cD_{a^+}^{\mu,\psi} x_{n+1}(r) \\ &= f(r, w(r)) - f(r, x_n(r)) + m(x_{n+1}(r) - x_n(r)) \\ &\ge -m(w(r) - x_n(r)) + m(x_{n+1}(r) - x_n(r)) \end{aligned}$$

$$= -m(w(r) - x_{n+1}(r))$$

$$= -mp(r),$$

$$\text{and } p(a) = 0,\ p(b) = 0.$$

Then by Lemma 4.8, we obtain $p(r) \geq 0$, $r \in J$ implies that $x_{n+1}(r) \leq w(r)$, $r \in J$. Set $p(r) = y_{n+1}(r) - w(r)$. We have

$$\begin{aligned}
{}^cD_{a^+}^{\mu,\psi} p(r) &= {}^cD_{a^+}^{\mu,\psi} y_{n+1}(r) - {}^cD_{a^+}^{\mu,\psi} w(r) \\
&= f(r, y_n(r)) - m(y_{n+1}(r) - y_n(r)) - f(r, w(r)) \\
&\geq -m(y_n(r) - w(r)) - m(y_{n+1}(r) - y_n(r)) \\
&= -m(y_{n+1}(r) - w(r)) \\
&= -mp(r),
\end{aligned}$$

$$\text{and } p(a) = 0,\ p(b) = 0.$$

By Lemma 4.8, we obtain $p(r) \geq 0$, $r \in J$. Thus $w(r) \leq y_{n+1}(r)$, $r \in J$. Hence, we have

$$x_{n+1}(r) \leq w(r) \leq y_{n+1}(r), r \in J. \tag{4.3.11}$$

Taking the limit as $n \to \infty$ on both sides of equation (4.3.11), we get

$$x_* \leq w \leq y_*.$$

Therefore x_*, y_* are the minimal and maximal solutions of nonlinear BVP (4.1.1) in $[x_0, y_0]$. □

In the following Theorem, we establish uniqueness of solution of nonlinear BVP (4.1.1).

Theorem 4.12. *Assume that $[H_1]$, $[H_2]$ are satisfied.*

$[H_3]$ *There exists a constant $m_* \geq -m$ such that*

$$f(r, y) - f(r, x) \leq m_*(y - x),$$

for every $x_0 \leq x \leq y \leq y_0$, $r \in J$. Then nonlinear BVP (4.1.1) has a unique solution in $[x_0, y_0]$.

Proof. By Theorem 4.11, x_* and y_* are respectively minimal and maximal solutions of the nonlinear BVP (4.1.1) and $x_* \leq y_*$, $r \in J$. It is sufficient to prove that $x_* \geq y_*$, $r \in J$. For this set $p(r) = x_* - y_*$, $r \in J$. In view of $[H_3]$, we have

$$\begin{aligned} {}^cD_{a^+}^{\mu,\psi} p(r) &= {}^cD_{a^+}^{\mu,\psi} x_* - {}^cD_{a^+}^{\mu,\psi} y_* \\ &= f(r, x_*) - f(r, y_*) \\ &\geq m_*(x_* - y_*) = m_* p(r). \end{aligned}$$

Furthermore, $p(a) = x_*(a) - y_*(a) = a^* - a^* = 0$ and $p(b) = x_*(b) - y_*(b) = b^* - b^* = 0$.

By Lemma 4.8, we obtain $p(r) \geq 0$, $r \in J$ implies that $x_* \geq y_*$.

Thus $x_* = y_*$ is unique solution of the nonlinear BVP (4.1.1) in $[x_0, y_0]$. □

4.4 Conclusion

1. We start with lower solution $x_0(r)$ and upper solution $y_0(r)$ as initial iterations of the nonlinear problem (4.1.1), the two monotone convergent sequences $\{x_n(r)\}$ and $\{y_n(r)\}$ are constructed.

2. Monotone technique coupled with lower and upper solutions of the nonlinear BVP (4.1.1) involving ψ-Caputo fractional derivative is developed.

3. Monotone technique is applied successfully to obtain existence and uniqueness of solutions of the problem (4.1.1) with boundary conditions.

Chapter 5

Nonlinear System of Initial Value Problem with R-L Fractional Derivative

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5.1 Introduction

This chapter deals with the study of existence and uniqueness of solution of the following nonlinear system with initial conditions:

$$(D^{2q}u_i)(t) = f_i(t,\, u_1,\, u_2, D^q u_1, D^q u_2),\ t \in (0, T] \tag{5.1.1}$$

$$t^{1-q}u_i(t)|_{t=0} = u_i^0, t^{1-q}(D^q u_i)(t)|_{t=0} = u_i^1,\ \ i = 1,\ 2. \tag{5.1.2}$$

where $0 < T < \infty$, u_i^0, u_i^1 are constants and $f_i \in C([0,\, T] \times \mathbb{R}^4)$, is quasimonotone nondecreasing, D^q is the standard Riemann- Liouville fractional derivative of order $0 < q \leq 1$. We organize the chapter as follows. In section 5.2, preliminary definitions and some basic results are considered. Some important lemmas and comparison results are also given. In section 5.3, we develop monotone technique for system of IVP with Riemann-Liouville fractional derivative. Existence and uniqueness of solutions of the nonlinear system of IVP are obtained.

5.2 Preliminaries

In this section, we deduce some preliminary results that will be used in the next section to attain existence and uniqueness results for the nonlinear system of initial value problem (5.1.1)- (5.1.2). Assume that $J = [0,\, T] \subset \mathbb{R}$ is a compact interval and

$$f_i(t,\, u_1(t),\, u_2(t),\, D^q u_1(t),\, D^q u_2(t)) \in C(J \times \mathbb{R}^4,\, \mathbb{R}),$$

is quasimonotone nondecreasing. Let

$$C([0,\, T]) = \big\{u_i | u_i(t) \text{ is continuous on } [0,\, T],\ \|u_i\|_C = \max_{t\in[0,\,T]} |u_i(t)|\big\},$$

$$C_{1-q}([0,\, T]) = \big\{u_i \in C([0,\, T]) : t^{1-q} u_i(t) \in C([0,\, T]),$$
$$|u_i\|_{C_{1-q}} = \|t^{1-q} u_i(t)\|_C\big\},$$

$$C^q_{1-q}([0,\, T]) = \{u_i \in C_{1-q}([0,\, T]) : t^{1-q} D^q u_i(t) \in C([0,\, T])\}.$$

Definition 5.1. A function $f_i(t,\, u_1(t),\, u_2(t),\, D^q u_1(t),\, D^q u_2(t)) \in C(J \times \mathbb{R}^4,\, \mathbb{R})$, $i = 1, 2,\ J = [0,\, T]$ is said to be quasimonotone nondecreasing (nonincreasing) if for each i, $u_i \le v_i$ and $u_j = v_j$, $i \ne j$, then

$$f_i(t,\, u_1(t),\, u_2(t),\, D^q u_1,\, D^q u_2) \le f_i(t,\, v_1(t),\, v_2(t),\, D^q v_1,\, D^q v_2)$$
$$\left(f_i(t,\, u_1(t),\, u_2(t),\, D^q u_1,\, D^q u_2) \ge f_i(t,\, v_1(t),\, v_2(t),\, D^q v_1,\, D^q v_2)\right).$$

Definition 5.2. A function $v_i^0 = (v_1^0,\, v_2^0) \in C^q_{1-q}([0,\, T])$ is called a lower solution of IVP(5.1.1)- (5.1.2) if it satisfies

$$(D^{2q} v_i^0)(t) \le f_i(t,\, v_1(t),\, v_2(t),\, D^q v_1(t),\, D^q v_2(t)), t \in (0,\, T]$$
$$t^{1-q} v_i^0(t)|_{t=0} \le v_i^0,\ \ t^{1-q} D^q v_i^0(t)|_{t=0} \le v_i^1.$$

Definition 5.3. A function $w_i^0 = (w_1^0,\, w_2^0) \in C_{1-q}^q([0,\, T])$ is called a upper solution of IVP(5.1.1)-(5.1.2), if it satisfies

$$(D^{2q}w_i^0)(t) \geq f_i(t,\, w_1(t),\, w_2(t),\, D^q w_1(t),\, D^q w_2(t)),\;\; t \in (0,\, T]$$

$$t^{1-q}w_i^0(t)|_{t=0} \geq w_i^0,\;\; t^{1-q}D^q w_i^0(t)|_{t=0} \geq w_i^1.$$

Definition 5.4. The sector denoted by Ω is defined as

$$\Omega = \left[v_i^0,\, w_i^0\right] = \Big\{u_i \in C_{1-q}^q([0,\, T]) : v_i^0 \leq u_i \leq w_i^0,\, t \in [0,\, T];$$
$$t^{1-q}v_i^0(t)|_{t=0} \leq t^{1-q}u_i(t)|_{t=0} \leq t^{1-q}w_i^0(t)|_{t=0},$$
$$t^{1-q}D^q v_i^0(t)|_{t=0} \leq t^{1-q}D^q u_i(t)|_{t=0} \leq t^{1-q}D^q w_i^0(t)|_{t=0}\Big\}.$$

Definition 5.5. Let $f_i : J \times \mathbb{R}^4 \to \mathbb{R}$ be a real valued continuous function. We say that $f_i(t, u_1(t), u_2(t), D^q u_1(t), D^q u_2(t))$ satisfies one sided Lipschitz condition, if there exist constants $M_i,\; N_i \in \mathbb{R}$, $N_i^2 > 4M_i$, such that, for $v_i^0 \leq v_i \leq w_i \leq w_i^0$,

$$f_i(t, w_1, \ldots, D^q w_2) - f_i(t, v_1, \ldots, D^q v_2) \geq -N_i(D^q w_i - D^q v_i) - M_i(w_i - v_i). \tag{5.2.1}$$

Further to ensure the uniqueness of solution of IVP (5.1.1) – (5.1.2), there exist $M_i, N_i \in \mathbb{R}$, $N_i^2 > 4M_i$, such that, for $v_i^0 \leq v_i \leq w_i \leq w_i^0$,

$$f_i(t, w_1, \ldots, D^q w_2) - f_i(t, v_1, \ldots, D^q v_2) \leq N_i(D^q w_i - D^q v_i) +$$

$$M_i(w_i - v_i). \qquad (5.2.2)$$

From conditions (5.2.1) and (5.2.2), we conclude that the function f_i satisfies Lipschitz condition if there exists constants $N_i,\ M_i \geq 0$, $N_i^2 > 4M_i$ such that

$$|f_i(t, w_1, \dots, D^q w_2) - f_i(t, v_1, \dots, D^q v_2)| \leq N_i|D^q w_i - D^q v_i| + M_i|w_i - v_i|. \qquad (5.2.3)$$

Now, we consider the following result for the linear fractional initial value problem to obtain existence and uniqueness results of solution of the IVP (5.1.1) − (5.1.2).

Lemma 5.6. *[71] Suppose that $u(t) \in C_{1-q}([0, T])$, then the linear initial value problem*

$$D^q u(t) + Mu(t) = \sigma(t), \quad t^{1-q}u(t)|_{t=0} = u_0,\, t \in (0, T],$$

where $M \in \mathbb{R}$ and $\sigma(t) \in C_{1-q}([0, T])$, has the following integral representation of solution $u(t) = \Gamma(q)u_0 e_q(-Mt) + [e_q(-Mx) * \sigma(x)](t)$, where $(g * f)(t) = \int_0^t g(t-x)f(x)\,dx$, and

$$e_q(\lambda z) = z^{q-1}E_{q,q}(\lambda z^q) = z^{q-1}\sum_{k=0}^{\infty} \lambda^k \frac{z^{qk}}{\Gamma((k+1)q)},$$

where $E_{q,q} = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma((k+1)q)}$, *is Mittag-Leffler function of two parameter.*

Lemma 5.7. *[71] Suppose that* $u(t) \in C^q_{1-q}([0, T])$ *then the linear initial value problem*

$$(D^{2q}u)(t) + N(D^q u)(t) + Mu(t) = \sigma(t), \quad t \in (0, T]$$
$$t^{1-q}u(t)|_{t=0} = u_0, \tag{5.2.4}$$
$$t^{1-q}D^q u(t)|_{t=0} = u_1,$$

where $M, N \in \mathbb{R}$ *are constants,* $N^2 > 4M$ *and* $\sigma(t) \in C_{1-q}([0, T])$, *has the following representation of solution*

$$u(t) = \Gamma(q)u_0 e_q(\lambda_2 t) + \Gamma(q)(u_1 - \lambda_2 u_0)[e_q(\lambda_2 x) * e_q(\lambda_1 x)](t)$$
$$+ [e_q(\lambda_2 x) * e_q(\lambda_1 x) * \sigma(x)](t), \tag{5.2.5}$$

where $\lambda_1 = \frac{-N+\sqrt{N^2-4M}}{2}$, $\lambda_2 = \frac{-N-\sqrt{N^2-4M}}{2} \leq 0$.

Lemma 5.8. *[71] The following result holds:*

$$[e_q(\lambda_2 x) * e_q(\lambda_1 x)](t) = [e_q(\lambda_1 x) * e_q(\lambda_2 x)](t)$$
$$= \frac{1}{\lambda_1 - \lambda_2}[e_q(\lambda_1 x) - e_q(\lambda_2 x)](t).$$

Lemma 5.9 (Comparison result)**.** *[71] If* $u(t) \in C_{1-q}([0, T])$ *and satisfies the relation* $D^q u(t) + Mu(t) \geq 0, t^{1-q}u(t)|_{t=0} \geq 0, \quad t \in$

$(0, T]$, *where,* $M \in \mathbb{R}$ *is constant. Then* $u(t) \geq 0,\ t \in (0, T]$.

Lemma 5.10 (Comparison result)**.** *[71] If* $u(t) \in C^q_{1-q}([0, T])$ *and satisfies* $(D^{2q}u)(t) + N(D^q u)(t) + Mu(t) = \sigma(t) \geq 0,\ t \in (0, T]$, $t^{1-q}u(t)|_{t=0} = u_0 \geq 0$, $t^{1-q}D^q u(t)|_{t=0} = u_1 \geq 0$, *where,* $N, M \in \mathbb{R}$, $N^2 > 4M$ *are constants such that*

$$\lambda_1 = \frac{-N + \sqrt{N^2 - 4M}}{2} \geq 0 > \lambda_2 = \frac{-N - \sqrt{N^2 - 4M}}{2}.$$

Then $u(t) \geq 0,\ t \in (0, T]$.

5.3 Monotone Method

In this section, we prove the existence and uniqueness theorem of solution for IVP (5.1.1) – (5.1.2).

Theorem 5.11. *Assume that*

(i) $v_i^0 = (v_1^0, v_2^0)$ *and* $w_i^0 = (w_1^0, w_2^0)$ *in* $C^q_{1-q}([0, T])$ *are ordered lower and upper solutions of IVP (5.1.1) – (5.1.2) respectively.*

(ii) $f_i \equiv f_i(t, u_1, u_2, D^q u_1, D^q u_2) \in C(J \times \mathbb{R}^4, \mathbb{R}), J = [0, T]$ *satisfies one-sided Lipschitz condition,* $i = 1, 2$.

(iii) $f_i \equiv f_i(t, u_1, u_2, D^q u_1, D^q u_2)$ *are quasi-monotone non-decreasing*

then there exist monotone sequences $\{v_i^n(t)\}$ *and* $\{w_i^n(t)\}$ *such that* $\{v_i^n\} \to v_i(t), \quad \{w_i^n\} \to w_i(t)$ *as* $n \to \infty$, *where* $v_i(t)$ *and* $w_i(t)$ *are minimal and maximal solutions on the ordered interval* $[v_i^0, w_i^0]$ *of IVP* (5.1.1) – (5.1.2) *and satisfy the monotone property*

$$v_i^0 \le v_i^1 \le v_i^2 \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0, \quad i = 1, 2.$$

Proof. For any $\eta_i(t) = (\eta_1, \eta_2) \in \Omega$. Consider the linear initial value problem

$$\begin{aligned}(D^{2q}u_i)(t) &= f_i(t,\, \eta_1,\, \eta_2,\, D^q\eta_1,\, D^q\eta_2) + N_i(D^q\eta_i - D^q u_i) + M_i(\eta_i - u_i)\\ &= \sigma(\eta_i),\\ t^{1-q}u_i(t)|_{t=0} &= u_i^0,\ t^{1-q}(D^q u_i)(t)|_{t=0} = u_i^1,\ i = 1, 2. \end{aligned} \tag{5.3.1}$$

It is clear that, by Lemma 5.7 and (5.2.3), linear initial value problem (5.3.1) has exactly one solution $u_i \in C_{1-q}^q([0,\, T])$ and whose integral representation is as in (5.2.5). Now define

$$\begin{aligned} u_i(t) &= A[\eta_i,\, \mu]\\ &= \Gamma(q)u_i^0 e_q(\lambda_2^i t) + \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)[e_q(\lambda_2^i x) * e_q(\lambda_1^i x)](t) +\\ &\qquad [e_q(\lambda_2^i x) * e_q(\lambda_1^i x) * \sigma(\eta_i)(x)](t) \end{aligned}$$

where , $\lambda_1^i = \frac{-N_i+\sqrt{N_i^2-4M_i}}{2} \geq 0 > \lambda_2^i = \frac{-N_i-\sqrt{N_i^2-4M_i}}{2}$. For each $\eta_i(t) = (\eta_1, \eta_2)$ and $\mu_i(t) = (\mu_1, \mu_2)$ in Ω such that

$$v_i^0(0) \leq \eta_i(t) \leq \mu_i(t) \leq w_i^0(0).$$

Define an operator A from $[v_i^0,\, w_i^0]$ into $C_{1-q}^q([0,\, T])$ and η_i is solution of the IVP (5.1.1)-(5.1.2) if and only if $\eta_i = A[\eta_i, \mu]$ and μ_i is solution of the IVP (5.1.1)-(5.1.2) if and only if $\mu_i = A[\eta, \mu_i]$. First we prove that,

(a) $v_i^0 \leq A[v_i^0,\, w_i^0],\ w_i^0 \geq A[w_i^0,\, v_i^0]$,

(b) A possesses the monotone property on the segment $\Omega = [v_i^0,\, w_i^0]$.

To prove (a), set $A[v_i^0,\, w_i^0] = v_i^1(t) = (v_1^1, v_2^1)$, where $v_i^1(t)$ is the unique solution of system (5.3.1) and set $p_i(t) = v_i^0(t) - v_i^1(t)$ with $\eta_i = v_i^0(t)$. Observe that

$$\begin{aligned} D^{2q}p_i(t) &= D^{2q}v_i^0(t) - D^{2q}v_i^1(t) \\ &\leq f_i(t, v_1^0, v_2^0, D^q v_1^0, D^q v_2^0) - f_i(t, v_1^0, v_2^0, D^q v_1^0, D^q v_2^0) \\ &\qquad - N_i(D^q v_i^0 - D^q v_i^1) - M_i(v_i^0 - v_i^1), \\ &= -N_i(D^q v_i^0 - D^q v_i^1) - M_i(v_i^0 - v_i^1) \\ &= -N_i(D^q p_i)(t) - M_i(p_i)(t) \end{aligned}$$

$$\textit{Thus} \quad D^{2q}p_i(t) \leq -N_i(D^q p_i)(t) - M_i(p_i)(t),$$

and

$$t^{1-q}p_i(t)|_{t=0} = t^{1-q}v_i^0(t)|_{t=0} - t^{1-q}v_i^1(t)|_{t=0} \le 0,$$

$$t^{1-q}(D^q p_i)(t)|_{t=0} = t^{1-q}(D^q v_i^0)(t)|_{t=0} - t^{1-q}(D^q v^1)_i(t)|_{t=0} \le 0.$$

By Lemma 5.10, we have

$$p_i(t) \le 0 \Rightarrow v_i^0(t) - v_i^1(t) \le 0$$

$$\Rightarrow v_i^0(t) \le v_i^1(t) = A[v_i^0, w_i^0]$$

To prove that $w_i^0 \ge A[w_i^0, v_i^0]$, set, $A[w_i^0, v_i^0] = w_i^1 = (w_1^1, w_2^1)$ is the unique solution of system (5.3.1). Set $p_i(t) = w_i^0 - w_i^1$ with $\eta_i = w_i^0(t)$.

$$\begin{aligned} D^{2q}p_i(t) &= D^{2q}w_i^0(t) - D^{2q}w_i^1(t) \\ &\ge f_i(t, w_1^0, w_2^0, D^q w_1^0, D^q w_2^0) - N_i(D^q w_i^0 - D^q w_i^1) \\ &\quad - f_i(t, w_1^0, w_2^0, D^q w_1^0, D^q w_2^0) - M_i(w_i^0 - w_i^1) \\ &= -N_i(D^q w_i^0 - D^q w_i^1) - M_i(w_i^0 - w_i^1) \end{aligned}$$

$$\textit{Thus} \quad D^{2q}p_i(t) \ge -N_i(D^q p_i) - M_i(p_i)$$

$$t^{1-q}p_i(t)|_{t=0} = t^{1-q}w_i^0(t)|_{t=0} - t^{1-q}w_i^1(t)|_{t=0} \ge 0,$$

$$t^{1-q}(D^q p_i)(t)|_{t=0} = t^{1-q}(D^q w_i^0)(t)|_{t=0} - t^{1-q}(D^q w^1)_i(t)|_{t=0} \ge 0.$$

Then by Lemma 5.10,

$$p_i(t) \geq 0 \Rightarrow w_i^0(t) - w_i^1(t) \geq 0 \Rightarrow w_i^0(t) \geq w_i^1(t) = A[w_i^0, v_i^0].$$

Now to prove (b), if $v_i^0 \leq \eta_i \leq \mu_i \leq w_i^0$, then prove $A[\eta_i, \mu] \leq A[\eta, \mu_i]$ where $A[\eta_i, \mu] = u_i = (u_i^1, u_i^2)$ and $A[\eta, \mu_i] = v_i = (v_i^1, v_i^2)$. Consider $p_i(t) = u_i(t) - v_i(t)$ then observe that,

$$\begin{aligned}
(D^{2q}p_i)(t) &= (D^{2q}u_i)(t) - (D^{2q}v_i)(t) \\
&= f_i(t, \eta_1, \eta_2, D^q\eta_1, D^q\eta_2) - f_i(t, \mu_1, \mu_2, D^q\mu_1, D^q\mu_2) + \\
&\qquad N_i(D^q\eta_i - D^q u_i) + M_i(\eta_i - u_i) - \\
&\qquad N_i(D^q\mu_i - D^q v_i) - M_i(\mu_i - v_i) \\
&\leq N_i(D^q\mu_i - D^q\eta_i) + M_i(\mu_i - \eta_i) + N_i(D^q\eta_i - D^q u_i) + \\
&\qquad M_i(\eta_i - u_i) - N_i(D^q\mu_i - D^q v_i) - M_i(\mu_i - v_i) \\
&= N_i(D^q v_i - D^q u_i) + M_i(v_i - u_i)
\end{aligned}$$

Thus $\quad (D^{2q}p_i)(t) \leq -N_i(D^q p_i)(t) - M_i(p_i)(t)$

$$t^{1-q}p_i(t)|_{t=0} = t^{1-q}u_i(t)|_{t=0} - t^{1-q}v_i(t)|_{t=0} = u_i^0 - v_i^0 \leq 0,$$

$$t^{1-q}(D^q p_i)(t)|_{t=0} = t^{1-q}(D^q u_i)(t)|_{t=0} - t^{1-q}(D^q v_i)(t)|_{t=0} = u_i^1 - v_i^1 \leq 0.$$

By Lemma 5.10, $p_i(t) \leq 0 \Rightarrow u_i(t) \leq v_i(t)$. Hence, $A[\eta_i, \mu] \leq A[\eta, \mu_i]$. Thus the operator A possesses the monotone property on $\Omega = [v_i^0, w_i^0]$. Define the sequences $\{v_i^n(t)\}$ and $\{w_i^n(t)\}$ by

$v_i^n = A[v_i^{n-1}, w_i^{n-1}]$ and $w_i^n = A[w_i^{n-1}, v_i^{n-1}]$. Then, we obtain,

$v_i^0 \le v_i^1 \le v_i^2 \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0$.

Let $P_i = \{v_i^n : n \in \mathbb{N}\}$, $Q_i = \{w_i^n : n \in \mathbb{N}\}$. We show that the set P_i, Q_i are relatively compact in $C^q_{1-q}([0,\, T])$. For any $\eta_i(t) \in \Omega$ and by definition of lower and upper solution and Lipschitz condition, we have

$$\begin{aligned}
(D^{2q}v_i^0)(t) + N_i(D^q v_i^0)(t) + M_i v_i^0(t) &\le f_i(t,\, v_1^0,\, v_2^0,\, D^q v_1^0,\, D^q v_2^0) + \\
&\quad N_i(D^q v_i^0)(t) + M_i(v_i^0)(t) \\
&\le f_i(t,\, \eta_1,\, \eta_2,\, D^q\eta_1,\, D^q\eta_2) + N_i(D^q\eta_i)(t) + M_i(\eta_i)(t) \\
&\le f_i(t\,, w_1^0,\, w_2^0,\, D^q w_1^0,\, D^q w_2^0) + N_i(D^q w_i^0)(t) + M_i(w_i^0)(t) \\
&\le (D^{2q}w_i^0)(t) + N_i(D^q w_i^0)(t) + M_i w_i^0(t).
\end{aligned}$$

Let P_i, Ω in $C^q_{1-q}([0,\, T])$ be bounded sets. Furthermore

$$\sigma_i(\eta_i(t)) = f_i(t,\, \eta_1,\, \eta_2,\, D^q\eta_1,\, D^q\eta_2) + N_i(D^q\eta_i)(t) + M_i(\eta_i)(t) : \eta_i \in \Omega$$

is also bounded set. Hence, there exist a constant $L_i > 0$ such that

$$\begin{aligned}
||\sigma_i(v_i^n)(t)|| &= \max_{0 \le t \le T} |t^{1-q}\sigma_i(v_i^n)(t)| \le L_i \\
\Leftrightarrow |\sigma_i(v_i^n)(t)| &\le L_i t^{1-q}, t \in (0,\, T]. \qquad (5.3.2)
\end{aligned}$$

On the other hand $\{v_i^n : n \in \mathbb{N}\}$ satisfy

$$v_i^n = \Gamma(q)u_0 e_q(\lambda_2^i t) + \Gamma(q)(u_1^i - \lambda_2^i u_0^i)[e_q(\lambda_2^i x) * e_q(\lambda_1^i x)](t) \\ + [e_q(\lambda_2^i x) * e_q(\lambda_1^i x) * \sigma(v_i^{n-1})](t). \quad (5.3.3)$$

Let $G(\lambda_j^i, t) = t^{1-q}[e_q(\lambda_j^i t) * \sigma(v_i^{n-1})](t)], \ t \in (0, T]$. Without loss of generality, assume that $0 \le t_1 \le t_2 \le T$. Since $\lambda_2^i < 0 \le \lambda_1^i$, we have

$$|G(\lambda_2^i, t_1) - G(\lambda_2^i, t_2)| \le \frac{L_i \Gamma(q)}{|\lambda_1^i|}|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| + \\ \frac{2L_i \Gamma(q)}{\Gamma(2q)}(t_2 - t_1)^q, \quad (5.3.4)$$

$$|G(\lambda_1^i, t_1) - G(\lambda_1^i, t_2)| \le (\frac{L_i \Gamma(q)}{|\lambda_1^i|} + \frac{L_i T^q}{q})|E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| \\ + \frac{2L_i \Gamma(q)}{\Gamma(2q)} E_{q,q}(\lambda_1^i T^q)(t_2 - t_1)^q. \quad (5.3.5)$$

From $E_{q,q}(t) \in C([0, T])$ and $\forall \epsilon > 0 \ \exists \ \delta = \delta(\epsilon)$, when $|t_2 - t_1| < \delta$, we have

$$|E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| < \frac{\epsilon}{6L_i^1}, \quad (5.3.6)$$

$$|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| < \frac{\epsilon}{6L_i^2}, \quad (5.3.7)$$

$$(t_2 - t_1)^q < \frac{\epsilon}{6L_i^3}, \tag{5.3.8}$$

where

$$L_i^1 = max\left\{\frac{\Gamma(q)|(u_i^1 - \lambda_2^i u_i^0)\lambda_1^i|}{|\lambda_1^i - \lambda_2^i|}, \frac{L_i}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}[\Gamma^2(q) + \frac{|\lambda_1^i|T^q}{q}]\right\},$$

$$L_i^2 = max\bigg\{\Gamma(q)|u_i^0|, \frac{\Gamma(q)|(u_i^1 - \lambda_2^i u_i^0)\lambda_1^i|}{|\lambda_1^i - \lambda_2^i|},$$
$$\frac{L_i}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}[\Gamma^2(q) + \frac{|\lambda_1^i|T^q}{q}]\bigg\},$$

$$L_i^3 = \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}[1 + E_{q,q}(\lambda_1^i T^q)].$$

Using (5.3.4) to (5.3.8), we obtain

$$|t_1^{1-q}v_i^n(t_1) - t_2^{1-q}v_i^n(t_2)| = \bigg|\Gamma(q)u_i^0\left[t_1^{1-q}e_q(\lambda_2^i t_1) - t_2^{1-q}e_q(\lambda_2^i t_2)\right]$$
$$+ \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\bigg[t_1^{1-q}e_q(\lambda_2^i t_1) * e_q(\lambda_1^i t_1) -$$
$$t_2^{1-q}e_q(\lambda_2^i t_2) * e_q(\lambda_1^i t_2)\bigg]$$
$$+ \bigg[t_1^{1-q}e_q(\lambda_2^i t_1) * e_q(\lambda_1^i t_1) * \sigma(v_i^{n-1})(t_1)$$
$$- t_2^{1-q}e_q(\lambda_2^i t_2) * e_q(\lambda_1^i t_2) * \sigma(v_i^{n-1})(t_2)\bigg]\bigg|$$

$$= \bigg|\Gamma(q)u_i^0[E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)]$$
$$+ \frac{\Gamma(q)(u_i^1 - \lambda_2^i u_i^0)}{\lambda_1^i - \lambda_2^i}\bigg[(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)) +$$
$$(E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q))\bigg]$$
$$+ \frac{1}{\lambda_1^i - \lambda_2^i}\bigg\{[t_1^{1-q}e_q(\lambda_2^i t_1) * \sigma(v_i^{n-1})(t_1) -$$

$$t_2^{1-q} e_q(\lambda_2^i t_2) * \sigma(v_i^{n-1})(t_2)]$$

$$+ [t_2^{1-q} e_q(\lambda_1^i t_2) * \sigma(v_i^{n-1})(t_2) - t_1^{1-q} e_q(\lambda_1^i t_1) * \sigma(v_i^{n-1})(t_1)] \Big\} \Big|,$$

$$\leq \Gamma(q) \left|u_i^0\right| |E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)|$$

$$+ \frac{\Gamma(q)|u_i^1 - \lambda_2^i u_i^0|}{|\lambda_1^i - \lambda_2^i|} \Big[|(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q))| +$$

$$|(E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| \Big]$$

$$+ \frac{1}{|\lambda_1^i - \lambda_2^i|} |t_1^{1-q} e_q(\lambda_2^i t_1) * \sigma(v_i^{n-1})(t_1) -$$

$$t_2^{1-q} e_q(\lambda_2^i t_2) * \sigma(v_i^{n-1})(t_2)|$$

$$+ \frac{1}{|\lambda_1^i - \lambda_2^i|} |t_1^{1-q} e_q(\lambda_1^i t_1) * \sigma(v_i^{n-1})(t_1) -$$

$$t_2^{1-q} e_q(\lambda_1^i t_2) * \sigma(v_i^{n-1})(t_2)]|,$$

$$\leq \Gamma(q)|u_i^0| \left[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| \right]$$

$$+ \frac{\Gamma(q)|u_i^1 - \lambda_2^i u_i^0|}{|\lambda_1^i - \lambda_2^i|} \Big[|(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q))| +$$

$$|(E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| \Big]$$

$$+ \frac{L_i \Gamma(q)}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|} \Big[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| +$$

$$\frac{2L_i \Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|} (t_2 - t_1)^q \Big]$$

$$+ \frac{1}{|\lambda_1^i - \lambda_2^i|} \left[\frac{L_i \Gamma(q)}{|\lambda_1^i|} + \frac{L_i T^q}{q} \right] \left[|E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)| \right]$$

$$+ \frac{2L_i \Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|} E_{q,q}(\lambda_1^i T^q)(t_2 - t_1)^q,$$

$$
\begin{aligned}
&\leq \Gamma(q)|u_i^0| \left[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)|\right] + \\
&\quad \frac{\Gamma(q)|u_i^1 - \lambda_2^i u_i^0|}{|\lambda_1^i - \lambda_2^i|}\Bigg[|(E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q))| + \\
&\qquad\qquad |E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\Bigg] \\
&\quad + \frac{L_i\Gamma(q)}{|\lambda_1^i - \lambda_2^i||\lambda_1^i|}\left[\Gamma(q) + (\Gamma(q) + \frac{\lambda_1^i T^q}{q})\right] \\
&\qquad \left[|E_{q,q}(\lambda_2^i t_1^q) - E_{q,q}(\lambda_2^i t_2^q)| + |E_{q,q}(\lambda_1^i t_1^q) - E_{q,q}(\lambda_1^i t_2^q)|\right] \\
&\quad + \frac{2L_i\Gamma(q)}{\Gamma(2q)|\lambda_1^i - \lambda_2^i|}\left[1 + E_{q,q}(\lambda_1^i T^q)\right](t_2 - t_1)^q, \\
&< \epsilon.
\end{aligned}
$$

Thus P_i is equi-continuous. Then by Ascoli- Arzela theorem, we conclude that P_i is relatively compact set of $C_{1-q}^q([0, T])$. Similarly we can show that Q_i is relatively compact set of $C_{1-q}^q([0, T])$. Therefore the sequences $\{v_i^n(t)\}$,$\{w_i^n(t)\}$ converges uniformly to $v_i(t), w_i(t)$ respectively on $[0, T]$. Then we have point-wise limits

$\lim_{n\to\infty} v_i^n(t) = v_i(t), \ \lim_{n\to\infty} w_i^n(t) = w_i(t),$

$\lim_{n\to\infty} D^q v_i^n(t) = D^q v_i(t), \ \lim_{n\to\infty} D^q w_i^n(t) = D^q w_i(t) \ \ t \in (0, T].$

Thus by relations $(v_i^0 \leq v_i^1 \leq v_i^2 \leq \ldots)$, it follows that $v_i(t)$ and $w_i(t)$ satisfy the following monotone property

$$v_i^0 \leq v_i^1 \leq \ldots \leq v_i^n \leq v_i \leq w_i \leq w_i^n \leq \ldots \leq w_i^1 \leq w_i^0,$$

$$D^q v_i^0 \leq D^q v_i^1 \leq \ldots \leq D^q v_i^n \leq \ldots \leq D^q w_i^n \leq \ldots \leq D^q w_i^1 \leq D^q w_i^0.$$

Now, we prove that $v_i(t)$, $w_i(t)$ are respectively minimal and maximal solutions of initial value problem (5.1.1) − (5.1.2). Since $f_i(i = 1, 2)$ is continuous then clearly the function $\sigma(\eta_i(t))$ is continuous and monotone nondecreasing in $v_i(t)$ implies that $\{\sigma(v_i^n(t))\}$ converges to $\sigma(v_i)(t)$, $t \in (0, T]$. Taking limit as $n \to \infty$ of $\{v_i^n(t)\}$ and using dominated convergence theorem, $v_i(t)$ satisfies the integral equation

$$v_i(t) = \Gamma(q)u_i^0 e_q(\lambda_2^i t) + \Gamma(q)(u_i^1 - \lambda_2^i u_i^0)\left[e_q(\lambda_2^i x) * e_q(\lambda_1^i x)\right](t) +$$

$$\left[e_q(\lambda_2^i x * e_q(\lambda_1^i x)\sigma(v_i)(x)\right](t).$$

Thus $v_i(t)$ is an integral representation of the solution of IVP (5.1.1)− (5.1.2). By the assumption of the function $f_i(i = 1, 2)$ and Lemma 5.7, it follows that $v_i(t)$ is a classical solution of IVP (5.1.1) − (5.1.2). This proves that the lower sequence $\{v_i^n(t)\}$ converges to a solution $v_i(t)$ of IVP (5.1.1) − (5.1.2). Similarly, we can prove that the upper sequence $\{w_i^n(t)\}$ converges to a solution $w_i(t)$ of IVP (5.1.1)−(5.1.2) and satisfies the relation $v_i(t) \le w_i(t), i = 1, 2, t \in (0, T]$. It follows that $v_i^0 \le v_i^1 \le \ldots \le v_i^n \le v_i \le w_i \le w_i^n \le \ldots \le w_i^1 \le w_i^0$, holds as well as $v_i(t)$ and $w_i(t)$ are minimal and maximal solution of IVP (5.1.1) − (5.1.2) in the sector Ω. □

Now we prove uniqueness of solution of the IVP (5.1.1) − (5.1.2) in the following:

Theorem 5.12. *Assume that*

(i) v_i^0 *and* w_i^0 *in* C_{1-q}^q *are ordered lower and upper solutions of IVP* $(5.1.1) - (5.1.2)$,

(ii) $f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_1) \in C(J \times \mathbb{R}^4, \mathbb{R})$ *is quasimonotone non-decreasing,*

(iii) $f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_1)$ *satisfies both sided Lipschitz condition.*

Then the IVP $(5.1.1) - (5.1.2)$ *has unique solution in the sector* Ω.

Proof. Observe that

$$-N_i(D^q u_i - D^q u_i^*) - M_i(u_i - u_i^*) \le f_i(t,\, u_1,\, u_2,\, D^q u_1,\, D^q u_2) - f_i(t,\, u_1^*,\, u_2^*,\, D^q u_1^*,\, D^q u_2^*)$$
$$\le N_i(D^q u_i - D^q u_i^*) + M_i(u_i - u_i^*)$$

for $v_i^0 \le u_i^* \le u_i \le w_i^0$ which follows from (5.2.3). Then the Theorem 5.11 implies that the initial value problem $(5.1.1) - (5.1.2)$ has unique solution in sector $[v_i^0,\, w_i^0]$. □

5.4 Conclusion

1. Starting with lower and upper solutions $v^0(t)$ and $w^0(t)$ as initial iterations of the problem $(5.1.1) - (5.1.2)$ when the function on

the right is quasi monotone nondecreasing, the two monotone convergent sequences $\{v^n(t)\}$ and $\{w^n(t)\}$ are constructed.

2. Monotone iterative technique with lower and upper solutions is developed for the nonlinear system (5.1.1) − (5.1.2) involving Riemann -Liouville fractional derivative.

3. Monotone iterative technique is applied successfully to prove the existence and uniqueness of solutions of the problem (5.1.1) − (5.1.2) with initial conditions.

Chapter 6

Nonlinear System of Initial Value Problem with ψ-Caputo Fractional Derivative

The content of this chapter is communicated in the Scopus indexed journal as :

Nonlinear Coupled System of Initial Value Problems Involving ψ-Caputo Fractional Derivative

6.1 Introduction

In 2020, Derbazi et al. [20] developed monotone iterative technique to study the existence and uniqueness of solution for initial value problem [IVP]of nonlinear fractional differential equations including ψ-Caputo derivative. In chapter 4, we studied qualitative properties of solutions of nonlinear boundary value problem involving ψ-Caputo fractional derivative.

Motivated by their works, we consider in this chapter, the following system of nonlinear ψ-Caputo fractional differential equations with initial conditions

$$
\begin{aligned}
{}^{c}D_{p^+}^{\mu,\psi} w_i(r) &= F_i(r, w_1(r), w_2(r)), \quad r \in \mathscr{J} = [p, q], \qquad (6.1.1)\\
w_i(p) &= c_i^*,
\end{aligned}
$$

where ${}^{c}D_{p^+}^{\mu,\psi}$ is $\psi-$Caputo fractional derivative of order $\mu, 0 < \mu \leq 1$, $F_i \in C(\mathscr{J} \times \mathbb{R}^2, \mathbb{R})$, $c_i^* \in \mathbb{R}, i = 1, 2$. We develop monotone technique and prove the existence and uniqueness of solution of system (6.1.1).

The rest of the chapter is organized as follows: In section 6.2, basic definitions and useful lemmas are given. In section 6.3, lower-upper solutions of IVP for ψ-Caputo fractional differential equation are introduced. Monotone technique is developed and successfully applied to obtain existence and uniqueness of solution of nonlinear system (6.1.1).

6.2 Preliminaries

In this section, we deduce some preliminary results required in the next section to attain existence and uniqueness results for nonlinear system (6.1.1) involving ψ-Caputo fractional derivative. Let $\mathscr{J} = [p, q]$, where $0 \leq p < q < \infty$, be a finite interval and $\psi : \mathscr{J} \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(r) \neq 0$, for all $r \in \mathscr{J}$.

Definition 6.1. A pair of functions $y^0 = (y_1^0, y_2^0)$ and $z^0 = (z_1^0, z_2^0)$ in $C(\mathscr{J}, R)$ with $(y_1^0, y_2^0) \leq (z_1^0, z_2^0)$ are called ordered lower and upper solutions of system (6.1.1) if

$$\begin{aligned} {}^cD_{p^+}^{\mu,\psi} y_i^0(r) &\leq F_i(r, y_1^0(r), y_2^0(r)), \quad y_i^0(p) \leq c_i^*, \\ \text{and} \quad {}^cD_{p^+}^{\mu,\psi} z_i^0(r) &\geq F_i(r, z_1^0(r), z_2^0(r)), \quad z_i^0(p) \geq c_i^*. \end{aligned}$$

Define the sector:

$$\Omega = \left\langle y^0,\, z^0 \right\rangle = \left\{ w_i \in C(\mathscr{J}, R) : y^0 \leq (w_1, w_2) \leq z^0,\, r \in \mathscr{J} \right\}.$$

Definition 6.2. A function $F_i(r, w_1(r), w_2(r)) \in C(\mathscr{J} \times \mathbb{R}^2, \mathbb{R})$, $i, j = 1, 2$ is said to be quasi-monotone non-decreasing if for each i, $w_i \leq x_i$ and $w_j = x_j$, $i \neq j$, then $F_i(r, w_1(r), w_2(r)) \leq F_i(r, x_1(r), x_2(r))$.

Definition 6.3. Let $F_i : \mathscr{J} \times \mathbb{R}^2 \to \mathbb{R}$ be a real valued continuous function. We say that $F_i(r,\, w_1(r),\, w_2(r))$ satisfies one sided Lipschitz condition, if there exist $m_i \in \mathbb{R}$ such that

$$F_i(r,\, w_1(r),\, w_2(r)) - F_i(r,\, x_1(r),\, x_2(r)) \geq -m_i(w_i(r) - x_i(r)) \tag{6.2.1}$$

$y^0 \leq x_i \leq w_i \leq z^0, \quad i = 1, 2.$

Now consider the following lemmas, which play very important role in further development.

Lemma 6.4. *[20] The initial value problem for ψ-Caputo fractional differential equation*

$$
\begin{aligned}
{}^{c}D_{p^+}^{\mu,\psi} w(r) + Pw(r) = F(r), \quad r \in \mathscr{J},\\
w(p) = c^*,
\end{aligned}
$$

has unique solution

$$
\begin{aligned}
w(r) = {} & c^* E_{\mu,1}(-P(\psi(r) - \psi(p))^{\mu}) + \\
& \int_p^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1} E_{\mu,\mu}(-P(\psi(r) - \psi(p))^{\mu}) F(s)\, ds,
\end{aligned}
$$

and $E_{\mu,\nu}(.)$ is the two-parameter Mittag-Leffler function.

Lemma 6.5. *(Comparison Result). [20] Let $\mu \in (0,1]$ and $P \in \mathbb{R}$. If $w \in C(\mathscr{J}, \mathbb{R})$ satisfies the following inequalities*

$$
\begin{aligned}
{}^{c}D^{\mu,\psi}_{p^+} w(r) &\geq -Pw(r),\ r \in \mathscr{J}, \\
w(p) &\geq 0,
\end{aligned}
$$

then $w(r) \geq 0$ for all $r \in \mathscr{J}$.

6.3 Main Results

In this section, we develop monotone iterative technique for the non-linear system (6.1.1) and we prove the existence of solution for non-linear system (6.1.1).

Theorem 6.6. *Assume that,*

(i) $y^0 = (y_1^0, y_2^0)$ and $z^0 = (z_1^0, z_2^0)$ in $C(\mathscr{J}, \mathbb{R})$ are ordered lower and upper solutions of system (6.1.1) respectively,

(ii) $F_i(r, w_1, w_2) \in C(\mathscr{J} \times \mathbb{R}^2, \mathbb{R}), i = 1, 2,$ satisfies one-sided Lipschitz condition (6.2.1),

(iii) $F_i(r, w_1, w_2), i = 1, 2$ is quasi-monotone non-decreasing,

then there exist monotone sequences $\{y_i^n\} = \{y_1^n, y_2^n\}$ *and* $\{z_i^n\} = \{z_1^n, z_2^n\}$ *such that,*

$$\{y_i^n\} \to y_i(r) = (y_1(r), y_2(r)),\ \ \{z_i^n\} \to z_i(r) = (z_1(r), z_2(r)), as\, n \to \infty$$

where $y_i(r)$ *and* $z_i(r)$*are minimal and maximal solution of nonlinear system (6.1.1) and satisfy the monotone property*

$$y_i^0 \leq y_i^1 \leq y_i^2 \leq \ldots \leq y_i^n \leq z_i^n \leq \ldots \leq z_i^1 \leq z_i^0, i = 1, 2.$$

Proof. For any $\eta(r) = (\eta_1, \eta_2) \in \Omega$ such that $y_1^0 \leq \eta_1, y_2^0 \leq \eta_2$ on $\mathscr{J}$, where $y_i^0 \leq \eta_i \leq z_i^0$.

We consider the system of linear fractional initial value problem

$$\begin{aligned} {}^c D_{p^+}^{\mu,\psi} w_i^1(r) &= F_i(r,\, \eta_1,\, \eta_2) - m_i(w_i^1 - \eta_i), \\ w_i^1(p) &= c_i^*. \end{aligned} \tag{6.3.1}$$

Then by Lemma 6.4, linear problem (6.3.1) have unique solution $w^1(r) = (w_1^1(r), w_2^1(r))$ on $\mathscr{J}$. For each $\eta(r)$ and $\mu(r)$ in $C(\mathscr{J}, \mathbb{R})$ satisfying $y_i^0(r) \leq \eta_i(r)$ and $y_i^0(r) \leq \mu_i(r)$,

define an operator A by $A[\eta, \mu] = w^1(r)$. Now, we prove the following:

$[D_1]$ $y^0 \leq A[y^0, z^0]$ and $z^0 \geq A[z^0, y^0]$,

$[D_2]$ A possesses the monotone property on the sector Ω.

To prove $[D_1]$, set $A[y^0, z^0] = y^1(r)$, where $y^1(r) = (y_1^1, y_2^1)$ is unique solutions of linear fractional initial value problem (6.3.1) with $\eta_i(r) = y_i^0(r)$. Setting $q_i(r) = y_i^0(r) - y_i^1(r)$, we see that

$$\begin{aligned}
{}^cD_{p^+}^{\mu,\psi} q_i(r) &= {}^cD_{p^+}^{\mu,\psi} y_i^0 - {}^cD_{p^+}^{\mu,\psi} y_i^1, \\
&\leq F_i(r,\, y_1^0,\, y_2^0) - F_i(r,\, y_1^0,\, y_2^0) + m_i(y_i^1 - y_i^0), \\
&= -m_i q_i(r), \\
\therefore \quad {}^cD_{p^+}^{\mu,\psi} q_i(r) &\leq -m_i q_i(r), \\
\text{and} \quad q_i(p) &= y_i^0(p) - y_i^1(p) \leq c_i^* - c_i^* = 0.
\end{aligned}$$

By Lemma 6.5 $\quad q_i(r) \leq 0 \Rightarrow y_i^0(r) \leq y_i^1(r),$

which implies $y_i^0 \leq A[y^0, z^0]$. Note that y_i^1 is unique solution of IVP (6.3.1) respectively.

Set $A[z^0, y^0] = z^1(r)$, where $z^1(r) = (z_1^1, z_2^1)$ is unique solution of system of linear fractional IVP (6.3.1) with $\mu_i(r) = z_i^0(r)$. Setting $q_i(r) = z_i^0(r) - z_i^1(r)$, then we see that

$$\begin{aligned}
{}^cD_{p^+}^{\mu,\psi} q_i(r) &= {}^cD_{p^+}^{\mu,\psi} z_i^0 - {}^cD_{p^+}^{\mu,\psi} z_i^1, \\
&\geq F_i(r,\, z_1^0,\, z_2^0) - F_i(r,\, z_1^0,\, z_2^0) + m_i(z_i^1 - z_i^0) \\
&= -m_i q_i(r)
\end{aligned}$$

$$\text{and} \quad q_i(p) = z_i^0(p) - z_i^1(p)$$
$$\geq c_i^* - c_i^* = 0.$$
$$\text{By Lemma 6.5} \quad q_i(r) \geq 0 \Rightarrow z_i^0(r) \geq z_i^1(r),$$

which implies $z_i^0 \geq A[z^0, y^0]$. Note that z_i^1 is unique solution of IVP (6.3.1) respectively. This proves $[D_1]$.

To prove $[D_2]$, let $\eta(r) = (\eta_1, \eta_2)$, $\beta(r) = (\beta_1, \beta_2)$ and $\mu(r) = (\mu_1, \mu_2)$ in Ω with $\eta(r) \leq \beta(r)$.

Suppose that, $A[\eta, \mu] = y_i^1(r)$ and $A[\beta, \mu] = z_i^1(r)$. Set $q_i(r) = y_i^1(r) - z_i^1(r)$, we observe that

$$\begin{aligned} {}^cD_{p^+}^{\mu,\psi} q_i(r) &= {}^cD_{p^+}^{\mu,\psi} y_i^1 - {}^cD_{p^+}^{\mu,\psi} z_i^1, \\ &= F_i(r,\, \eta_1,\, \eta_2) - F_i(r,\, \mu_1,\, \mu_2) + m_i(\eta_i - y_i^1) - \\ &\qquad m_i(\mu_i - z_i^1), \\ &\leq m_i(\mu_i - \eta_i) + m_i(\eta_i - y_i^1) - m_i(\mu_i - z_i^1), \\ &= -m_i(y_i^1 - z_i^1) \\ &= -m_i q_i(r), \end{aligned}$$
$$\text{and} \quad q_i(p) = y_i^1(c) - z_i^1(p) = c_i^* - c_i^* = 0.$$
$$\text{By Lemma 6.5,} \quad q_i(r) \leq 0 \Rightarrow y_i^1(r) \leq z_i^1(r).$$

Hence $A[\eta, \mu] \leq A[\beta, \mu]$. Similarly, we can prove that $A[\eta, \nu] \leq$

$A[\eta, \mu]$. Thus the operator A possesses monotone property on Ω. Now in view of $[D_1]$ and $[D_2]$, define the sequences $y_i^n(r) = A[y_i^{n-1}, z_i^{n-1}]$, $z_i^n(r) = A[z_i^{n-1}, y_i^{n-1}]$ on Ω by

$$^{c}D_{p^+}^{\mu,\psi} y_i^n(r) = F_i(r,\, y_1^{n-1},\, y_2^{n-1}) - m_i\left(y_i^n - y_i^{n-1}\right), \quad y_i^n(p) = c_i^*,$$

$$^{c}D_{p^+}^{\mu,\psi} z_i^n(r) = F_i(r,\, z_1^{n-1},\, z_2^{n-1}) - m_i\left(z_i^n - z_i^{n-1}\right), \quad z_i^n(p) = c_i^*.$$

From $[D_1]$, we have $y_i^0(r) \leq y_i^1(r)$, $z_i^0(r) \geq z_i^1(r)$.

Now assume that $y_i^{k-1}(r) \leq y_i^k(r)$, $z_i^{k-1}(r) \geq z_i^k(r)$ and we claim that $y_i^k(r) \leq y_i^{k+1}(r)$, $z_i^k(r) \geq z_i^{k+1}(r)$ and $y_i^k(r) \leq z_i^k(r)$. To prove this, set $q_i(r) = y_i^k(r) - y_i^{k+1}(r)$, and observe that

$$\begin{aligned}
^{c}D_{p^+}^{\mu,\psi} q_i(r) &= {}^{c}D_{p^+}^{\mu,\psi} y_i^k - {}^{c}D_{p^+}^{\mu,\psi} y_i^{k+1}, \\
&= F_i(r,\, y_1^{k-1},\, y_2^{k-1}) - m_i\left(y_i^k - y_i^{k-1}\right) - \\
&\quad F_i(r,\, y_1^k,\, y_2^k) + m_i\left(y_i^{k+1} - y_i^k\right), \\
&\leq m_i\left(y_i^k - y_i^{k-1}\right) - m_i\left(y_i^k - y_i^{k-1}\right) + \\
&\quad m_i\left(y_i^{k+1} - y_i^k\right), \\
&= -m_i(y_i^k(r) - y_i^{k+1}(r)) \\
&= -m_i q_i(r),
\end{aligned}$$

$$\text{and} \quad q_i(p) = y_i^k(c) - y_i^{k+1}(p) = c_i^* - c_i^* = 0.$$

By Lemma 6.5, $\quad q_i(r) \leq 0 \Rightarrow y_i^k(r) \leq y_i^{k+1}(r)$.

Similarly, set $q_i(r) = z_i^{k+1}(r) - z_i^k(r)$, we observe that

$$\begin{aligned}
{}^cD_{p^+}^{\mu,\psi} q_i(r) &= {}^cD_{p^+}^{\mu,\psi} z_i^{k+1} - {}^cD_{p^+}^{\mu,\psi} z_i^k, \\
&= F_i(r, z_1^k, z_2^k) - m_i\left(z_i^{k+1} - z_i^k\right) - \\
&\qquad F_i(r, z_1^{k-1}, z_2^{k-1}) + m_i\left(z_i^k - z_i^{k-1}\right), \\
&\leq m_i\left(z_i^{k-1} - z_i^k\right) - m_i\left(z_i^{k+1} - z_i^k\right) + \\
&\qquad m_i\left(z_i^k - z_i^{k-1}\right), \\
&= -m_i(z_i^{k+1} - z_i^k) \\
&= -m_i q_i(r), \\
\text{and} \quad q_i(p) &= z_i^{k+1}(p) - z_i^k(p) = c_i^* - c_i^* = 0.
\end{aligned}$$

By Lemma 6.5, $\quad q_i(r) \leq 0 \Rightarrow z_i^{k+1}(r) \leq z_i^k(r)$.

Similarly prove that, $y_i^k(r) \leq z_i^k(r)$. Thus from mathematical induction principle, we obtain

$$y_i^0 \leq y_i^1 \leq y_i^2 \leq \ldots \leq y_i^n \leq z_i^n \leq z_i^{n-1} \leq \ldots \leq z_i^1 \leq z_i^0, \; i = 1, 2.$$

Thus the sequences $\{y_i^n(r)\}$ is monotone nondecreasing and bounded above and $\{z_i^n(r)\}$ is monotone nonincreasing and bounded below. Hence pointwise limit exists and are given by

$$\lim_{n\to\infty} y_i^n(r) = y_i(r) \quad \text{and} \quad \lim_{n\to\infty} z_i^n(r) = z_i(r),$$

uniformly on $r \in \mathscr{J}$.

Using corresponding fractional Volterra integral equations

$$\begin{aligned}
y_i^n(r) &= c_i^* E_{\mu,1}(-m(\psi(r)-\psi(p))^\mu) \\
&\quad + \int_c^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^\mu \\
&\qquad [f(s,x_n(s)) + m x_n(s)]\, ds, \\
z_i^n(r) &= c_i^* E_{\mu,\mu}(-m(\psi(r)-\psi(p))^\mu) \\
&\quad + \int_p^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1} E_{\mu,\mu}(-m(\psi(r)-\psi(s))^\mu) \\
&\qquad [f(s,y_n(s)) + m y_n(s)]\, ds,
\end{aligned}$$

it follows that $y(r)$ and $z(r)$ are the solutions of system (6.3.1).

Finally we claim that $y(r)$ and $z(r)$ are the minimal and maximal solutions of system (6.3.1). Let $x(r)$ be any solution of (6.1.1) different from $y(r)$ and $z(r)$, so that there exists k such that $y_k^i \le x_i(r) \le z_k^i$ on $\mathscr{J}$. To prove $y_i^{k+1} \le x_i(r) \le z_i^{k+1}$, set $q_i(r) = x_i(r) - y_i^{k+1}(r)$, we observe that

$$\begin{aligned}
{}^c D_{p^+}^{\mu,\psi} q_i(r) &= {}^c D_{p^+}^{\mu,\psi} x_i - {}^c D_{p^+}^{\mu,\psi} y_i^{k+1}, \\
&= F_i(r,\, x_1,\, x_2) - F_i(r,\, y_1^k,\, y_2^k) - m_i \left(y_i^{k+1} - y_i^k\right), \\
&\ge -m_i \left(x_i - y_i^k\right) \\
&= -m_i q_i(r), \\
\text{and} \quad q_i(p) &= y_i^{k+1}(p) - x_i(p) = c_i^* - c_i^* = 0.
\end{aligned}$$

By Lemma 6.5, $\quad q_i(r) \geq 0 \Rightarrow y_i^{k+1}(r) \leq x_i(r).$

Since $y_i^0(r) \leq x_i(r)$ on $\mathscr{J}$, then by induction principle, it follows that $y_i^k(r) \leq x_i(r)$ for all k. Similarly, prove that $x_i(r) \leq z_i^k(r)$ on $\mathscr{J}$ for all k. Thus $y_i^n(r) \leq x_i(r) \leq z_i^n(r)$ on $\mathscr{J}$. Taking limit as $n \to \infty$, it follows that $y_i(r) \leq x_i(r) \leq z_i(r)$. □

The uniqueness of solution of the system (6.1.1) is proved in the following theorem

Theorem 6.7. *Assume that,*

(i) $y^0 = (y_1^0, y_2^0)$ *and* $z^0 = (z_1^0, z_2^0)$ *in* $C(\mathscr{J}, \mathbb{R})$ *are ordered lower and upper solutions of IVP (6.1.1) respectively,*

(ii) *function* $F_i(r, w_1, w_2) \in C(\mathscr{J} \times \mathbb{R}^2, \mathbb{R}), i = 1, 2,$ *satisfies Lipschitz condition, if there exist* $m_i \in \mathbb{R}$ *such that*

$$|F_i(r, w_1(r), w_2(r)) - F_i(r, x_1(r), x_2(r))| \leq -m_i\,|w_i(r) - x_i(r)| \tag{6.3.2}$$

$y^0 \leq x_i \leq w_i \leq z^0,\ i = 1, 2.$

(iii) $F_i(r, w_1, w_2), i = 1, 2$ *is quasi-monotone non-decreasing,*

Then the nonlinear system (6.1.1) has unique solution in the order interval.

Proof. Since $y_i(r) \leq z_i(r)$. We need to prove $y_i(r) \geq z_i(r)$. Setting $q_i(r) = z_i(r) - y_i(r)$, and

$$\begin{aligned} {}^{c}D^{\mu,\psi}_{p^+} q_i(r) &= {}^{c}D^{\mu,\psi}_{p^+} z_i - {}^{c}D^{\mu,\psi}_{p^+} y_i, \\ &= F_i(r, z_1, z_2) - F_i(r, y_1, y_2), \\ &\leq -m_i\,(z_i - y_i) \\ &= -m_i q_i(r), \\ \text{and} \quad q_i(p) &= z_i(p) - y_i(p) = c_i^* - c_i^* = 0. \\ \text{By Lemma 6.5,} \quad q_i(r) &\leq 0 \Rightarrow y_i(r) \geq z_i(r). \end{aligned}$$

Hence $y_i(r) = z_i(r)$ is the unique solution of system (6.1.1) in the order interval. □

6.4 Conclusion

1. Starting with lower and upper solutions $y^0(r)$ and $z^0(r)$ as initial iterations of the problem (6.1.1), the two convergent sequences $\{y^n(r)\}$ and $\{z^n(r)\}$ are constructed.

2. Monotone technique coupled with lower and upper solutions is developed for the problem (6.1.1) involving ψ-Caputo fractional derivative.

3. Developed monotone technique is successfully applied to prove the existence of minimal and maximal solutions of the problem (6.1.1) with initial conditions.

4. Uniqueness of solution of the problem (6.1.1) is also proved using developed technique.

Chapter 7

Coupled Nonlinear System Involving ψ-Caputo Derivative

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7.1 Introduction

In this chapter, we consider the existence results for following nonlinear system of boundary value problem involving ψ-Caputo fractional derivative and we establish a comparison result. As an application monotone iterative technique coupled with method of lower-upper solutions, we prove existence and uniqueness of solution of the boundary value problem.

$$\begin{aligned} {}^cD^{\mu,\psi}_{p^+}w(r) &= F(r, w(r), z(r)), \\ {}^cD^{\mu,\psi}_{p^+}z(r) &= G(r, z(r), w(r)), \quad r \in \mathscr{J} = [p, q], \\ w(p) = c_1^*,\ z(p) = c_2^*, &\qquad w(q) = d_1^*,\ z(q) = d_2^*, \end{aligned} \tag{7.1.1}$$

where ${}^cD^{\mu,\psi}_{p^+}$ is the $\psi-$Caputo fractional derivative of order μ, $0 < \mu \leq 1$, $F, G \in C(\mathscr{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $c_i^*, d_i^* \in \mathbb{R}, i = 1, 2$ and $c_1^* \leq c_2^*, d_1^* \leq d_2^*$.

7.2 Preliminaries

In this section, we consider some preliminary results required in the next section to obtain existence and uniqueness results for nonlinear boundary value problems (7.1.1) involving ψ-Caputo fractional derivative. Let $\mathscr{J} = [p, q]$, where $0 \leq p < q < \infty$, be a finite interval and $\psi : \mathscr{J} \to \mathbb{R}$ is an increasing differentiable function such that $\psi'(r) \neq 0$,for all $r \in \mathscr{J}$.

Lemma 7.1 (A particular Case of Lemma 4.6). *The linear boundary value problem for ψ-Caputo fractional differential equation*

$$\begin{aligned} {}^cD_{p^+}^{\mu,\psi} w(r) + Pw(r) = F(r), \quad r \in \mathscr{J}, \\ w(p) = c^*, \quad w(q) = d^*, \end{aligned} \tag{7.2.1}$$

has unique solution

$$\begin{aligned} w(r) = c^* E_{\mu,1}(-P(\psi(r) - \psi(p))^{\mu}) + \theta E_{\mu,1}(-P(\psi(r) - \psi(p))^{\mu}) \\ + \int_0^r \psi'(s)[\psi(r) - \psi(s)]^{\mu-1} E_{\mu,\mu}(-P(\psi(r) - \psi(p))^{\mu}) F(s)\, ds, \end{aligned} \tag{7.2.2}$$

$$\begin{aligned} \textit{where} \quad \theta = d^* + \frac{[F(p) - Pw(p)][\psi(q) - \psi(p)]^{\mu}}{\Gamma(\mu + 1)} \\ - \frac{1}{\Gamma(\mu)} \int_p^q \psi'(s)[\psi(q) - \psi(s)]^{\mu-1}[F(s) - Pw(s)]\, ds, \end{aligned}$$

and $E_{\mu,\nu}(.)$ is the two-parameter Mittag-Leffler function [30].

Lemma 7.2. *The linear boundary value problem for ψ-Caputo fractional differential equation*

$$\begin{aligned} {}^cD_{p^+}^{\mu,\psi} w(r) = F_1(r) - Pw(r) - Qz(r),\, r \in \mathscr{J}, \\ {}^cD_{p^+}^{\mu,\psi} z(r) = F_2(r) - Pz(r) - Qw(r),\, r \in \mathscr{J}, \\ w(p) = c_1^*,\ z(p) = c_2^*, \quad w(q) = d_1^*,\ z(q) = d_2^*, \end{aligned} \tag{7.2.3}$$

has unique system of solutions in $C(\mathscr{J},\mathbb{R})$.

Proof. The proof follows from the fact that the pair (w,z) is a solution of problem (7.2.3) if and only if $x(r)=w(r)+z(r)$, $y(r)=w(r)-z(r)$ for all $r\in\mathscr{J}$, where x,y are the solutions of the problem:

$$\begin{aligned}
{}^{c}D^{\mu,\psi}_{p^+}x(r) &= {}^{c}D^{\mu,\psi}_{p^+}w(r)+{}^{c}D^{\mu,\psi}_{p^+}z(r),\\
&=(F_1+F_2)(r)-(P+Q)\left[w(r)+z(r)\right],\\
&=(F_1+F_2)(r)-(P+Q)\left[x(r)\right], \qquad (7.2.4)\\
x(p) &= w(p)+z(p)=c_1^*+c_2^*,\\
x(q) &= w(q)+z(q)=d_1^*+d_2^*\\
{}^{c}D^{\mu,\psi}_{p^+}y(r) &= {}^{c}D^{\mu,\psi}_{c^+}w(r)-{}^{c}D^{\mu,\psi}_{c^+}z(r)\\
&=(F_1-F_2)(r)-(P-Q)\left[w(r)-z(r)\right],\\
&=(F_1-F_2)(r)-(P-Q)\left[y(r)\right], \qquad (7.2.5)\\
y(p) &= w(p)-z(p)=c_1^*-c_2^*,\\
y(q) &= w(q)-z(q)=d_1^*-d_2^*.
\end{aligned}$$

Then by Lemma 7.1, both problems (7.2.4), (7.2.5) have unique solutions,

$$x(r)=(c_1^*+c_2^*)E_{\mu,1}(-(P+Q)(\psi(r)-\psi(p))^{\mu})$$

$$+ \theta_1 E_{\mu,1}(-(P+Q)(\psi(r)-\psi(p))^{\mu})$$
$$+ \int_0^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}$$
$$E_{\mu,\mu}(-(P+Q)(\psi(r)-\psi(p))^{\mu})(F_1+F_2)(s)\,ds,$$

$$\text{where}\,\theta_1 = d_1^* + d_2^* +$$
$$\frac{[(F_1+F_2)(p)-(P+Q)w(p)][\psi(q)-\psi(p)]^{\mu}}{\Gamma(\mu+1)}$$
$$-\frac{1}{\Gamma(\mu)}\int_p^q \psi'(s)[\psi(q)-\psi(s)]^{\mu-1}$$
$$[(F_1+F_2)(s)-(P+Q)w(s)]\,ds,$$

$$\text{and} \quad y(r) = (c_1^* - c_2^*)E_{\mu,1}(-(P-Q)(\psi(r)-\psi(p))^{\mu})$$
$$+ \theta_2 E_{\mu,1}(-(P-Q)(\psi(r)-\psi(p))^{\mu})$$
$$+ \int_0^r \psi'(s)[\psi(r)-\psi(s)]^{\mu-1}$$
$$E_{\mu,\mu}(-(P-Q)(\psi(r)-\psi(p))^{\mu})(F_1-F_2)(s)\,ds,$$

$$\text{where} \quad \theta_2 = (d_1^* - d_2^*) +$$
$$\frac{[(F_1-F_2)(p)-(P-Q)w(p)][\psi(q)-\psi(p)]^{\mu}}{\Gamma(\mu+1)}$$
$$-\frac{1}{\Gamma(\mu)}\int_p^q \psi'(s)[\psi(q)-\psi(s)]^{\mu-1}$$
$$[(F_1-F_2)(s)-(P-Q)w(s)]\,ds,$$

respectively in $C(\mathscr{J},\mathbb{R})$. In consequence, $w(r)$ and $z(r)$ are unique.

□

Lemma 7.3. *(Comparison Result) [Chapter 4, Lemma 4.8] Let* $\mu \in (0,1]$ *and* $P \in \mathbb{R}$. *If* $w \in C(\mathscr{J}, \mathbb{R})$ *satisfies the following inequalities*

$$
\begin{aligned}
{}^{c}D_{p^+}^{\mu,\psi} w(r) &\geq -Pw(r),\ r \in \mathscr{J}, \\
w(p) &\geq 0, \quad w(q) \geq 0,
\end{aligned} \tag{7.2.6}
$$

then $w(r) \geq 0$ *for all* $r \in \mathscr{J}$.

Lemma 7.4. *(Comparison Result) Let* $\mu \in (0,1]$ *and* $P \in \mathbb{R}$ *and* $Q \geq 0$ *be given. If* $w, z \in C(\mathscr{J}, \mathbb{R})$ *satisfies the following inequalities*

$$
\begin{aligned}
{}^{c}D_{p^+}^{\mu,\psi} w(r) &\geq -Pw(r) + Qz(r),\ r \in \mathscr{J}, \\
{}^{c}D_{p^+}^{\mu,\psi} z(r) &\geq -Pz(r) + Qw(r),\ r \in \mathscr{J}, \\
w(p) \geq 0, z(p) \geq 0 &\quad w(q) \geq 0, z(q) \geq 0,
\end{aligned} \tag{7.2.7}
$$

then $w(r) \geq 0,\ z(r) \geq 0$ *for all* $r \in \mathscr{J}$.

Proof. Put $x(r) = w(r) + z(r)\ r \in \mathscr{J}$. Then by (7.2.7), we have

$$
\begin{aligned}
{}^{c}D_{p^+}^{\mu,\psi} x(r) &= {}^{c}D_{p^+}^{\mu,\psi} w(r) + {}^{c}D_{p^+}^{\mu,\psi} z(r), \\
&\geq -Pw(r) + Qz(r) - Pz(r) + Qw(r), \\
&= -(P - Q)(w(r) + z(r)) \\
&= -(P - Q)x(r), \\
x(p) &= w(p) + z(p) \geq 0,
\end{aligned}
$$

$$x(q) = w(q) + z(q) \geq 0.$$

Then by Lemma 7.3, $x(r) \geq 0 \Rightarrow w(r) + z(r) \geq 0$. Next to show that, $w(r) \geq 0$, $z(r) \geq 0$ for all $r \in \mathscr{J}$. By (7.2.7), we have

$$\begin{aligned}
{}^{c}D_{p^+}^{\mu,\psi} w(r) &\geq -Pw(r) + Qz(r), \\
&\geq -Pw(r) - Qw(r) = -(P+Q)w(r), \\
w(p) &\geq 0, \qquad w(q) \geq 0, \\
{}^{c}D_{p^+}^{\mu,\psi} z(r) &\geq -Pz(r) + Qw(r), \\
&\geq -Pz(r) - Qz(r) \\
&= -(P+Q)z(r), \\
z(p) &\geq 0, \qquad z(q) \geq 0.
\end{aligned}$$

Then by Lemma 7.3, it is easy to show that $w(r) \geq 0$, $z(r) \geq 0$ for all $r \in \mathscr{J}$. □

7.3 Existence and Uniqueness Results

In this section, we develop monotone iterative technique and apply it to prove the existence of minimal and maximal solutions of the nonlinear BVP (7.1.1) involving ψ-Caputo fractional derivative. We list the following assumptions for convenience.:

(B_1) There exist $w_0(r), z_0(r) \in C(\mathscr{J}, \mathbb{R})$ and $w_0(r) \leq z_0(r)$, such that

$$\begin{aligned} {}^cD_{p^+}^{\mu,\psi} w_0(r) &\leq F(r, w_0(r), z_0(r)), \\ w_0(p) \leq &c_1^*,\; w_0(q) \leq d_1^*, \\ {}^cD_{p^+}^{\mu,\psi} z_0(r) &\geq G(r, z_0(r), w_0(r)), \\ z_0(p) \geq &c_2^*,\; z_0(q) \geq d_2^*. \end{aligned}$$

(B_2) There exist $P \in \mathbb{R}$, $Q \geq 0$ such that

$$F(r, w(r), z(r)) - F(r, w^*(r), z^*(r)) \geq -P(w - w^*) - Q(z - z^*),$$
$$G(r, z(r), w(r)) - G(r, z^*(r), w^*(r)) \geq -P(z - z^*) - Q(w - w^*),$$

where $w_0(r) \leq w^*(r) \leq w(r) \leq z_0(r)$, $w_0(r) \leq z^*(r) \leq z(r) \leq z_0(r)$ and

$$G(r, z(r), w(r)) - F(r, w(r), z(r)) \geq P(w(r) - z(r)) + Q(z(r) - w(r)),$$

with $w_0(r) \leq w(r) \leq z(r) \leq z_0(r)$.

Theorem 7.5. *Suppose that the assumptions (B_1) and (B_2) hold, then there exist monotone iterative sequences $\{w_n\}$, $\{z_n\} \in [w_0, z_0]$ such that $w_n \to w_*$, $z_n \to z_*$ as $(n \to \infty)$ uniformly on $r \in \mathscr{J}$, where $(w_*, z_*) \in [w_0, z_0] \times [w_0, z_0]$ are minimal and maximal solutions of the* *nonlinear problem (7.1.1) and satisfy the monotone property*

$$w_0 \leq w_1 \leq \ldots \leq w_n \leq \ldots \leq w_* \leq z_* \leq \ldots \leq z_n \leq \ldots \leq z_1 \leq z_0. \tag{7.3.1}$$

Proof. For any $w_{n-1}(r), z_{n-1}(r) \in C(\mathscr{J}, \mathbb{R})$, $n \geq 1$, we consider the linear system

$$\begin{aligned} {}^cD^{\mu,\psi}_{p^+} w_n(r) &= F(r, w_{n-1}(r), z_{n-1}(r)) - P(w_n(r) - w_{n-1}(r)) \\ &\qquad - Q(z_n(r) - z_{n-1}(r)), \\ {}^cD^{\mu,\psi}_{p^+} z_n(r) &= G(r, z_{n-1}(r), w_{n-1}(r)) - P(z_n(r) - z_{n-1}(r)) \\ &\qquad - Q(w_n(r) - w_{n-1}(r)), \\ w_n(p) &= c_1^*,\ w_n(q) = d_1^*, \quad z_n(p) = c_2^*,\ z_n(q) = d_2^*. \end{aligned} \tag{7.3.2}$$

Then by Lemma 7.2, we know that system (7.3.2) has unique system of solutions in $C(\mathscr{J}, \mathbb{R})$. Next we show that $\{w_n\}$, $\{z_n\}$ satisfy the property

$$w_{n-1}(r) \leq w_n(r) \leq z_n(r) \leq z_{n-1}(r), n = 1, 2, \ldots.$$

First need to show that $w_0(r) \leq w_1(r) \leq z_1(r) \leq z_0(r)$.
Let $\phi(r) = w_1(r) - w_0(r)$, $\sigma(r) = z_0(r) - z_1(r)$. Then from (7.3.2) and (B_1) we have

$${}^cD^{\mu,\psi}_{p^+} \phi(r) = {}^cD^{\mu,\psi}_{p^+} w_1(r) - {}^cD^{\mu,\psi}_{p^+} w_0(r)$$

$$\geq F(r, w_0(r), z_0(r)) - P(w_1(r) - w_0(r)) -$$

$$Q(z_1(r) - z_0(r)) - F(r, w_0(r), z_0(r))$$

$$= -P(w_1(r) - w_0(r)) - Q(z_1(r) - z_0(r))$$

$$= -P\phi(r) + Q\sigma(r),$$

$$\text{and} \quad \phi(p) = w_1(p) - w_0(p) \geq c_1^* - c_1^* = 0,$$

$$\phi(q) = w_1(q) - w_0(q) \geq d_1^* - d_1^* = 0,$$

$${}^cD_{p^+}^{\mu,\psi}\sigma(r) = {}^cD_{p^+}^{\mu,\psi}z_0(r) - {}^cD_{p^+}^{\mu,\psi}z_1(r)$$

$$\geq G(r, z_0(r), w_0(r)) - G(r, z_0(r), w_0(r)) +$$

$$P(z_1(r) - z_0(r)) + Q(w_1(r) - w_0(r))$$

$$= P(z_1(r) - z_0(r)) + Q(w_1(r) - w_0(r))$$

$$= -P\sigma(r) + Q\phi(r)$$

$$\text{and} \quad \sigma(p) = z_0(p) - z_1(p) \geq d_2^* - d_2^* = 0,$$

$$\sigma(q) = z_0(q) - z_1(q) \geq d_2^* - d_2^* = 0.$$

Then by Lemma 7.4, we have $\phi(r) \geq 0$, $\sigma(r) \geq 0$ implies that $w_1(r) \geq w_0(r)$, $z_0(r) \geq z_1(r)$, $r \in \mathscr{J}$.

Let $\rho(r) = z_1(r) - w_1(r)$. Then from (7.3.2) and (B_2) we have

$${}^cD_{p^+}^{\mu,\psi}\rho(r) = {}^cD_{p^+}^{\mu,\psi}z_1(r) - {}^cD_{p^+}^{\mu,\psi}w_1(r)$$

$$= G(r, z_0(r), w_0(r)) - P(z_1(r) - z_0(r)) - Q(w_1 - w_0) -$$

$$- F(r, w_0(r), z_0(r)) + P(w_1(r) - w_0(r)) + Q(z_1 - z_0),$$

$$\geq P(w_0(r) - z_0(r)) + Q(z_0(r) - w_0(r)) - P(z_1 - z_0)$$

$$- Q(w_1(r) - w_0(r)) + P(w_1(r) - w_0(r)) + Q(z_1 - z_0),$$

$$= -(P - Q)(z_1(r) - w_1(r))$$

$$= -(P - Q)\rho(r),$$

and $\quad \rho(p) = z_1(p) - w_1(p) = c_2^* - c_1^* \geq 0,$

$$\rho(q) = z_1(q) - w_1(q) = d_2^* - d_1^* \geq 0.$$

Then by Lemma 7.3, we have $\rho(r) \geq 0$ implies that $z_1(r) \geq w_1(r)$, $r \in \mathscr{J}$. Hence we have $w_0(r) \leq w_1(r) \leq z_1(r) \leq z_0(r)$, $r \in \mathscr{J}$. Now,assume that $w_{i-1}(r) \leq w_i(r) \leq z_i(r) \leq z_{i-1}(r)$ for some $i \geq 1$. We claim that $w_i(r) \leq w_{i+1}(r) \leq z_{i+1}(r) \leq z_i(r)$ for some $r \in \mathscr{J}$. Set $\phi(r) = w_{i+1}(r) - w_i(r)$, $\sigma(r) = z_i(r) - z_{i+1}(r)$,
$\rho(r) = z_{i+1}(r) - w_{i+1}(r)$. By (7.3.2) and (B_2) we have

$${}^cD_{p^+}^{\mu,\psi}\phi(r) = {}^cD_{p^+}^{\mu,\psi}w_{i+1}(r) - {}^cD_{p^+}^{\mu,\psi}w_i(r),$$

$$\geq F(r, w_i(r), z_i(r)) - P(w_{i+1}(r) - w_i(r)) - Q(z_{i+1} - z_i)$$

$$- F(r, w_i(r), z_i(r)),$$

$$= -P(w_{i+1}(r) - w_i(r)) - Q(z_{i+1}(r) - z_i(r))$$

$$= -P\phi(r) + Q\sigma(r),$$

and $\quad \phi(p) = w_{i+1}(p) - w_i(p) = c_1^* - c_1^* = 0,$

$$\phi(q) = w_{i+1}(q) - w_i(q) = d_1^* - d_1^* = 0.$$

$$
\begin{aligned}
{}^{c}D^{\mu,\psi}_{p^+}\sigma(r) &= {}^{c}D^{\mu,\psi}_{p^+} z_i - {}^{c}D^{\mu,\psi}_{p^+} z_{i+1} \\
&\geq G(r, z_0(r), w_0(r)) - G(r, z_i(r), w_i(r)) + P(z_{i+1} - z_i) \\
&\qquad\qquad + Q(w_{i+1}(r) - w_i(r)) \\
&= P(z_{i+1}(r) - z_i(r)) + Q(w_{i+1}(r) - w_i(r)) \\
&= -P\sigma(r) + Q\phi(r), \\
\text{and} \quad \sigma(p) &= z_i(p) - z_{i+1}(p) = c_2^* - c_2^* = 0, \\
\sigma(q) &= z_i(q) - z_{i+1}(q) = d_2^* - d_2^* = 0.
\end{aligned}
$$

$$
\begin{aligned}
{}^{c}D^{\mu,\psi}_{p^+}\rho(r) &= {}^{c}D^{\mu,\psi}_{p^+} z_{i+1}(r) - {}^{c}D^{\mu,\psi}_{p^+} w_{i+1}(r) \\
&= G(r, z_i(r), w_i(r)) - P(z_{i+1}(r) - z_i(r)) - Q(w_{i+1} - w_i) \\
&\qquad - F(r, w_i(r), z_i(r)) + P(w_{i+1} - w_i) + Q(z_{i+1} - z_i) \\
&\geq P(w_i(r) - z_i(r)) + Q(z_i(r) - w_i(r)) - P(z_{i+1} - z_i) \\
&\qquad - Q(w_{i+1}(r) - w_i(r)) + P(w_{i+1} - w_i) + Q(z_{i+1} - z_i) \\
&= -P(z_{i+1}(r) - w_{i+1}(r)) + Q(z_{i+1}(r) - w_{i+1}(r)), \\
&= -(P - Q)(z_{i+1}(r) - w_{i+1}(r)) \\
&= -(P - Q)\rho(r),
\end{aligned}
$$

$$
\begin{aligned}
\text{and} \quad \rho(p) &= z_{i+1}(p) - w_{i+1}(p) = c_2^* - c_1^* \geq 0, \\
\rho(q) &= z_{i+1}(q) - w_{i+1}(q) = d_2^* - d_1^* \geq 0.
\end{aligned}
$$

Then by Lemma 7.3 and Lemma 7.4, we have

$$
\phi(r) \geq 0, \quad \sigma(r) \geq 0, \quad \rho(r) \geq 0.
$$

This implies that $w_i(r) \leq w_{i+1}(r) \leq z_{i+1}(r) \leq z_i(r)$ for some $r \in \mathscr{J}$.

Hence from the principle of mathematical induction, we have

$$w_0(r) \leq w_1(r) \leq \ldots \leq w_n(r) \leq \ldots \leq z_n(r) \leq \ldots \leq z_1(r) \leq z_0(r).$$

Thus the sequences $\{w_n(r)\}$ and $\{z_n(r)\}$ are uniformly bounded and convergent. Hence pointwise limit exists and are given by

$$\lim_{n\to\infty} w_n(r) = w_*(r) \quad \text{and} \quad \lim_{n\to\infty} z_n(r) = z_*(r),$$

uniformly on $r \in J$ and the limit functions w_*, z_* satisfy BVP (7.1.1).

Moreover, $w_*, z_* \in [w_0, z_0]$. Taking the limits in (7.3.2), we know that (w_*, z_*) is a system of solutions of (7.1.1) in $[w_0, z_0] \times [w_0, z_0]$. Moreover, (7.3.1) is true.

Finally, we prove that (7.1.1) has minimal and maximal solutions. Let $(w,\, z) \in [w_0, z_0] \times [w_0, z_0]$ is any system of solutions of (7.1.1). Assume that for some $n \in N$, $w_n \leq w, z \leq z_n$ for some $r \in \mathscr{J}$.

Set $\phi(r) = w(r) - w_{n+1}(r)$, $\sigma(r) = z_{n+1}(r) - z(r)$. Then by (7.3.2) and (B_2) we have

$$\begin{aligned}
{}^cD_{p^+}^{\mu,\psi}\phi(r) &= {}^cD_{p^+}^{\mu,\psi}w(r) - {}^cD_{p^+}^{\mu,\psi}w_{n+1}(r) \\
&= F(r, w(r), z(r)) - F(r, w_n, z_n) + P(w_{n+1}(r) - w_n(r)) \\
&\quad + Q(z_{n+1}(r) - z_n(r))
\end{aligned}$$

$$\geq -P(w(r) - w_n(r)) - Q(z(r) - z_n(r))$$

$$+ P(w_{n+1}(r) - w_n(r)) + Q(z_{n+1}(r) - z_n(r))$$

$$= -P(w(r) - w_n(r)) + Q(z_n(r) - z(r))$$

$$= -P\phi(r) + Q\sigma(r),$$

and $\quad \phi(p) = w(p) - w_{n+1}(p) = c_1^* - c_1^* = 0$

$$\phi(q) = w(q) - w_{n+1}(q) = d_1^* - d_1^* = 0.$$

$$^cD_{p^+}^{\mu,\psi}\sigma(r) = {}^cD_{p^+}^{\mu,\psi}z_{n+1}(r) - {}^cD_{p^+}^{\mu,\psi}z(r)$$

$$= G(r, z_n(r), w_n(r)) - P(z_{n+1}(r) - z_n(r))$$

$$- Q(w_{n+1}(r) - w_n(r)) - G(r, z(r), w(r))$$

$$\geq -P(z_n(r) - z(r)) - Q(w_n(r) - w(r))$$

$$- P(z_{n+1}(r) - z_n(r)) - Q(w_{n+1}(r) - w_n(r))$$

$$= -P(z_{n+1}(r) - z(r)) + Q(w(r) - w_{n+1}(r))$$

$$= -P\sigma(r) + Q\phi(r)$$

and $\quad \sigma(p) = z_{n+1}(p) - z(p) = c_2^* - c_2^* = 0,$

$$\sigma(q) = z_{n+1}(q) - z(q) = d_2^* - d_2^* = 0.$$

By Lemma 7.4, $w_{n+1}(r) \leq w(r), z(r) \leq z_{n+1}(r)$ for some $r \in \mathscr{J}$. Now taking the limits as $n \to \infty$, we have $w_* \leq w, z \leq z_*$. Hence (w_*, z_*) is an minimal and maximal solutions of system (7.1.1) in $[w_0, z_0] \times [w_0, z_0]$. This completes the proof. □

Example 7.1. *Consider the following coupled system of boundary values problem:*

$$
\begin{aligned}
{}^cD_{0+}^{1/2,r} w(r) &= -w^3(r) + 1 + z(r), \quad r \in J = [0,1], \\
{}^cD_{0+}^{1/2,r} z(r) &= -z^3(r) + 1 + w(r), \quad r \in J = [0,1], \qquad (7.3.3)\\
w(0) = z(0) = 0, &\quad w(1) = z(1) = \frac{3}{4}.
\end{aligned}
$$

Here $\mu = \frac{1}{2}$, $n = [\frac{1}{2}] + 1 = 1$ $c = 0$, $d = 1$, $c_1^* = c_2^* = 0$, $d_1^* = d_2^* = \frac{3}{4}$, $\psi(r) = r$. *Obviously* $F(r, w, z) = -w^3(r) + 1 + z(r)$, $G(r, z, w) = -z^3(r) + 1 + w(r)$.

Now taking $w_0(r) = 0$ *and* $z_0(r) = r$. *Also* $w_0{}_{\psi}^{[1]} = 0$ *and* $z_0{}_{\psi}^{[1]} = \frac{z_0'(r)}{\psi'(r)} = 1$ *then,* ${}^cD_{0+}^{1/2,r} w_0(r) = 0 \leq 1 + r = F(r, w_0, z_0)$, *and*

$$
\begin{aligned}
{}^cD_{0+}^{1/2,r} z_0(r) &= \frac{1}{\Gamma(\frac{1}{2})} \int_0^r (r-s)^{-1/2}\, ds \\
&= \frac{2}{\sqrt{\pi}} r^{1/2} \\
&\geq -r^3 + 1 \\
&= G(r, z_0, w_0).
\end{aligned}
$$

Also $w_0(r) \leq z_0(r)$. *Hence condition* (B_1) *of Theorem 7.5 holds. On the other hand, it is easy to show that for* $M = 3$ *and* $N = 0$, *condition* (B_2) *holds. Thus, all conditions of Theorem 7.5 are satisfied. Hence, the nonlinear system (7.3.3) has the minimal and maximal solution* $(w_*, z_*) \in [w_0, z_0] \times [w_0, z_0]$, *which can be obtained* *by taking limits from the iterative sequences, for* $n \geq 1$*,*

$$w_n(r) = \theta_1 E_{\frac{1}{2},1}(-3r^{\frac{1}{2}}) + \int_0^r (r-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-3r^{\frac{1}{2}})(-w_{n-1}^3(s) + 1 + z_{n-1}(s) + 3w_{n-1}(s))\, ds,$$

$$\text{where} \quad \theta_1 = \frac{3}{4} - \frac{1}{\Gamma(1/2)} \int_0^1 [1-s]^{-\frac{1}{2}} [-w_{n-1}^3 + 1 + z_{n-1} - 3w_{n-1}]\, ds,$$

$$z_n(r) = \theta_2 E_{\frac{1}{2},1}(-3r^{\frac{1}{2}}) + \int_0^r (r-s)^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}(-3r^{\frac{1}{2}})(-z_{n-1}^3(s) + 1 + w_{n-1}(s) + 3z_{n-1}(s))\, ds,$$

$$\text{where} \quad \theta_2 = \frac{3}{4} - \frac{1}{\Gamma(1/2)} \int_0^1 [1-s]^{-\frac{1}{2}} [-z_{n-1}^3 + 1 + w_{n-1} - 3z_{n-1}]\, ds.$$

7.4 Conclusion

1. Starting with lower and upper solutions $w_0(r)$ and $z_0(r)$ as initial iterations of the problem (7.1.1), the two monotone convergent sequences $\{w_n(r)\}$ and $\{z_n(r)\}$ are constructed.

2. Monotone technique coupled with lower and upper solutions is developed for the coupled system (7.1.1) involving ψ-Caputo fractional derivative.

3. Monotone technique is applied successfully to prove the existence of minimal and maximal solutions of the problem (7.1.1).

4. The obtained results are validated with suitable example.

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