CURRENT GLOBAL REVIEWER

Issue X (25) , Vol. II Dec. 2020	Peer Reviewed SJIF Impact factor	ISSN : 2319 - 8648 Impact Factor : 7.139

Impact Factor – 7.139 ISSN – 2319-8648

Curren Global Reviewer

Peer Reviewed Multidisciplinary International Research Journal

PEER REVIEWED & INDEXED JOURNAL

Dec.20 Issue- X Vol. II

Chief Editor

Mr. Arun B. Godam

Shaurya Publication , Latur

CURRENT GLOBAL REVIEWER

Issue X (25) , Vol. II | Peer Reviewed | ISSN : 2319 - 8648
Dec. 2020 | SJIF Impact factor | Impact Factor : 7.139

Index

CURRENT GLOBAL REVIEWER

Issue X (25) , Vol. II | Peer Reviewed | ISSN : 2319 - 8648
Dec. 2020 | SJIF Impact factor | Impact Factor : 7.139

Solutions for nonlinear Caputo fractional differential equations

Madhuri N. Gadsing
Department of Mathematics,Jawahar Arts, Science and Commerce College, Anadur, Tq.Tuljapur, Dist.Osmanabad, (M.S) India

---

Abstract

In this paper, nonlinear Caputo fractional differential equations is studied. The existence and uniqueness of a solution are investigated by using Krasnoselskii and Banach fixed point theorems and the method of lower-upper solutions.

Keywords: Existence and uniqueness solution, lower-upper solutions, fractional differential equations, fixed points.

Subject Classification Code: 34A08, 26A33, 47H10.

1 Introduction

Theory of fractional differential equations occur frequently in different research areas and engineering, such as Physics, Chemistry, Biology, medicine, aerodynamics, fields of control, electromagnetic etc.(see in [[4, 10, 9]] and references therein). In recent years, the theory of fractional differential equations has been given a great interest, especially to finding sufficient conditions for existence and uniqueness of the solutions of nonlinear fractional differential equations ([1, 2, 5, 6, 7, 8]).

In [[11]] S. Zhang investigated the existence and uniqueness of positive solutions for the nonlinear fractional differential equations by using the method of the upper and lower solution. In [[3]], Boulares et al. investigated existence and uniqueness of positive solutions for nonlinear fractional differential equations by using the method of the upper and lower solution and Banach fixed point theorem.

In this paper, using the method of upper and lower solutions and the Kransnoselskii and Banach fixed point theorems, we study the existence and uniqueness of solutions of the nonlinear fractional differential equation

$$\begin{cases} {}^cD^{\mu}z(r) = \mathscr{F}(r, z(r)) + {}^c D^{\mu-1}\mathscr{H}(r, z(r)), & 0 < r \leq 1, \\ z(0) = \zeta_1, \qquad z'(0) = \zeta_2 \geq \mathscr{H}(0, \zeta_1) > 0, \end{cases} \tag{1.1}$$

where $1 < \mu \leq 2$ and $\mathscr{F}, \mathscr{H} : [0, 1] \times [0, \infty) \to [0, \infty)$ are continuous functions and $\mathscr{H}$ is non-decreasing on z.

In the process we convert the given fractional differential equation into an equivalent integral equation. Then we construct appropriate mapping and employ Krasnoselskii fixed point theorem and the method of upper and lower solutions to show the existence of a solution of this equation. We also use the Banach fixed point theorem to show the existence of a unique positive solution.

Principal
Jawahar Arts, Science & Commerce College
Anadur, Tal. Tuljapur, Dist. Osmanabad.

CURRENT GLOBAL REVIEWER

Issue X (25) , Vol. II | Peer Reviewed | ISSN : 2319 - 8648
Dec. 2020 | SJIF Impact factor | Impact Factor : 7.139

2 Preliminaries

Let $B = C([0,1])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[0,1]$, endowed with the maximum norm. Let K be a nonempty closed subset of B defined as $K = \{z \in B : \|z\| \le l, l > 0\}$.

We give some definitions and their properties for our main results.

Definition 2.1 *The fractional integral of order $\mu > 0$ of a function $z : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$I^\mu z(r) = \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} z(s)\, ds.$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2 *The Caputo fractional derivative of order $\mu > 0$ of a function $u : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$^cD^\mu z(r) = \frac{1}{\Gamma(n-\mu)} \int_0^r (r-s)^{n-\mu-1} z^{(n)}(s)\, ds.$$

where $n = [\mu] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.1 ([4, 10]) *Let $Re(\mu) > 0$. suppose $z \in C^{n-1}([0, \infty])$ and $z^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$I^\mu(^cD_0^\mu z)(r) = z(r) - \sum_{k=0}^{n-1} \frac{z^{(k)}(0)}{k!} r^k.$$

In particular, when $1 < Re(\mu) < 2$, $I^\mu(^cD_0^\mu z)(r) = z(r) - z(0) - z'(0)r$.

Lemma 2.2 *Let $z \in C^1([0,1])$, z^2 and $\frac{\partial \mathscr{H}}{\partial r}$ exist, then $z(r)$ is a solution of (1.1) if and only if*

$$z(r) = \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1))r + \int_0^r \mathscr{H}(s, z(s))\, ds + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, z(s))\, ds \quad (2.1)$$

Proof: *Let z be a solution of (1.1). First we write equation-(1.1) as*

$$I^{\mu\, c}D^\mu z(r) = I^\mu \left(\mathscr{F}(r, z(r)) + {}^c D^{\mu-1} \mathscr{H}(r, z(r))\right), \quad 0 < r \le 1$$

Using lemma 2.1, we have

$$\begin{aligned} z(r) - z(0) - z'(0)r &= I^{\mu\, c}D^{\mu-1}\mathscr{H}(r, z(r)) + I^\mu \mathscr{F}(r, z(r)), \\ &= II^{\mu-1}[^cD^{\mu-1}\mathscr{H}(r, z(r))] + I^\mu \mathscr{F}(r, z(r)), \\ &= I(\mathscr{H}(r, w(r)) - \mathscr{H}(0, w(0))) + I^\mu \mathscr{F}(r, w(r)), \\ &= \int_0^r \mathscr{H}(s, w(s))\, ds - \mathscr{H}(0, \zeta_1)r \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, w(s))\, ds. \end{aligned}$$

then we obtain equation-(2.1). Conversely, suppose that $w(r)$ satisfies equation - (2.1). Then applying $^cD^\mu$ to both sides of equation -(2.1), we obtain $^cD^\mu z(r) = \mathscr{F}(r, z(r)) + {}^cD^{\mu-1}\mathscr{H}(r, z(r)), 0 < r \le 1$ and boundary conditions $z(0) = \zeta_1$, $z'(0) = \zeta_2 \ge \mathscr{H}(0, \zeta_1) > 0$ holds.

CURRENT GLOBAL REVIEWER

Issue X (25) , Vol. II Dec. 2020	Peer Reviewed SJIF Impact factor	ISSN : 2319 - 8648 Impact Factor : 7.139

Lastly, we state the fixed point theorems which enable us

Definition 2.3 *Let* $(B, \|.\|)$ *be a Banach space and* $\varphi : B \to B$. *The operator* φ *is a contraction operator if there is an* $\gamma \in (0,1)$ *such that* $u, v \in B$ *implying* $\|\varphi u - \phi v\| \leq \gamma \|u - v\|$

Theorem 2.3 [12], *Let* K *be a nonempty closed convex subset of a Banach space* B *and* $\varphi : K \to K$ *be a contraction operator. Then there is a unique* $z \in K$ *with* $\varphi z = z$.

Theorem 2.4 *(Krasnoselskii fixed point theorem)*([12]), *Let* K *be a nonempty closed convex subset of a Banach space* B *and let* P *and* Q *two operators defined on* K *with values in* B *such that* $Pu + Qv \in K$. *for every pair* $u, v \in K$, *the operator* P *is completely continuous and the operator* Q *is a contraction.Then there exist* $x \in K$ *such that* $x = Px + Qx$.

3 Existence and uniqueness of solutions

In this section, first we need to construct two mappings such as, one is contraction and other is completely continuous. First we define the operator $\varphi : K \to K$ such as

$$(\varphi z)(r) = \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1))r + \int_0^r \mathscr{H}(s, z(s))\, ds + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}(s, z(s))\, ds, \tag{3.1}$$

where the operator $\mathscr{P} : K \to B$ defined as,

$$(\mathscr{P}z)(r) = \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, z(s))\, ds,$$

and the operator $\mathscr{Q} : K \to X$ defined as,

$$(\mathscr{Q}Z)(r) = \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1))r + \int_0^r \mathscr{H}(s, z(s))\, ds.$$

Throughout this paper, we assume that the following condition holds

(C1) $\mathscr{H}, \mathscr{F} \in C([0,1] \times [0, \infty), [0, \infty))$ and $\mathscr{H}$ is non-decreasing on z.

(C2) For $u, v \in B$ and $r \in [0,1]$, there exist $\alpha \in (0,1)$ and $\beta > 0$ such that

$$|\mathscr{H}(r,u) - \mathscr{H}(r,v)| \leq \alpha |u - v|,$$

$$|\mathscr{F}(r,u) - \mathscr{F}(r,v)| \leq \beta |u - v|.$$

Lemma 3.1 *Assume that* [C1] *holds. Then the operator* $\mathscr{P} : K \to B$ *is completely continuous.*

Principal
Jawahar Arts, Science & Commerce College
Anadur, Tal. Tuljapur, Dist. Osmanabad.

Proof: By $[C1]$, $\mathscr{F}$ is continuous and nonnegative function, we get that $\mathscr{P}: K \to B$ is continuous. The function $\mathscr{F}: [0,1] \times [0,\infty) \to [0,\infty)$ is bounded then $\exists\, \lambda > 0$ such that $|\mathscr{F}(r, z(r))| \leq \lambda$. We obtain

$$\begin{aligned}|(\mathscr{P}z)(r)| &= \left|\frac{1}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}\mathscr{F}(s,z(s))\,ds\right|,\\ &\leq \frac{1}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}\,|\mathscr{F}(s,z(s))|\,ds,\\ &\leq \frac{\lambda}{\Gamma(\mu)}\left[\frac{(r-s)^\mu}{-\mu}\right]_0^r = \frac{\lambda r^\mu}{\Gamma(\mu+1)} \leq \frac{\lambda}{\Gamma(\mu+1)}.\end{aligned}$$

Thus $\|(\mathscr{P}z)(r)\| \leq \frac{\lambda}{\Gamma(\mu+1)}$. Hence $(Pw)(r)$ is uniformly bounded.
Now we will prove equicontinuity of $\mathscr{P}$. Let $z \in K$ and for any $r_1, r_2 \in [0,1]$ with $r_1 < r_2$ then we have

$$\begin{aligned}|(\mathscr{P}z)(r_1) - (\mathscr{P}z)(r_2)| &= \frac{1}{\Gamma(\mu)}\left|\int_0^{r_1}(r_1-s)^{\mu-1}\mathscr{F}(s,z(s))\,ds - \int_0^{r_2}(r_2-s)^{\mu-1}\mathscr{F}(s,z(s))\,ds\right|,\\ &\leq \frac{1}{\Gamma(\mu)}\int_0^{r_1}\left|(r_1-s)^{\mu-1} - (r_2-s)^{\mu-1}\right||\mathscr{F}(s,z(s))|\,ds\\ &+ \frac{1}{\Gamma(\mu)}\int_{r_1}^{r_2}\left|(r_2-s)^{\mu-1}\right||\mathscr{F}(s,z(s))|\,ds,\\ &\leq \frac{\lambda}{\Gamma(\mu)}\left[\int_0^{r_1}[(r_1-s)^{\mu-1} - (r_2-s)^{\mu-1}]\,ds + \int_{r_1}^{r_2}(r_2-s)^{\mu-1}\,ds\right],\\ &= \frac{\lambda}{\Gamma(\mu)}\left\{\left[\frac{(r_1-s)^\mu}{-\mu}\right]_0^{r_1} - \left[\frac{(r_2-s)^\mu}{-\mu}\right]_0^{r_1} + \left[\frac{(r_2-s)^\mu}{-\mu}\right]_{r_1}^{r_2}\right\},\\ &= \frac{\lambda}{\Gamma(\mu+1)}\{0 + (r_2-r_1)^\mu + r_1^\mu - r_2^\mu + (r_2-r_1)^\mu\},\\ &\leq \frac{2\lambda}{\Gamma(\mu+1)}(r_2-r_1)^\mu.\end{aligned}$$

Thus $|(z)(r_1) - (\mathscr{P}z)(r_2)| \to 0$ as $r_2 \to r_1$. Therefore $\mathscr{P}$ is equicontinuous on K. Then by Arzela-Ascoli theorem, $\mathscr{P}: K \to B$ is completely continuous.

Lemma 3.2 *Assume that $[C1]$ and $[C2]$ holds. Then the operator $\mathscr{Q}: K \to K$ is contraction.*

Proof: By $[C1]$ and boundary conditions of problem-(1.1), the operator $Q: K \to K$ is continuous. For $u, v \in K$, we have

$$\begin{aligned}|(\mathscr{Q}u)(r) - (\mathscr{Q}v)(r)| &= \left|\int_0^r \mathscr{H}(s,u(s))\,ds - \int_0^r \mathscr{H}(s,v(s))\,ds\right|\\ &= \int_0^r |\mathscr{H}(s,u(s)) - \mathscr{H}(s,v(s))|\,ds \leq \alpha r\,\|u-v\|.\end{aligned}$$

Thus

$$\|(\mathscr{Q}u)(r) - (\mathscr{Q}v)(r)\| \leq \alpha r\,\|u-v\|.$$

Hence Q is contraction.

Principal
Jawahar Arts, Science & Commerce College
Anadur, Tal. Tuljapur, Dist. Osmanabad

Let $c, d \in \mathbb{R}^+$ with $c < d$ and for any $z \in [c, d] \subset \mathbb{R}^+$, we define the upper and lower control functions respectively as follows

$$M(r, z) = \sup_{c \leq \eta \leq z} \mathscr{F}(r, \eta),$$

$$m(r, z) = \inf_{z \leq \eta \leq d} \mathscr{F}(r, \eta).$$

It is obvious that $m(r, z)$ and $M(r, z)$ are monotonic non-decreasing on $[c, d]$ and $m(r, z) \leq f(r, z) \leq M(r, z)$.

Definition 3.1 *Let z^*, $z_* \in K$, $c \leq z_* \leq z^* \leq d$ satisfying*

$$z^*(r) \geq \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1))r + \int_0^r \mathscr{H}(s, z^*(s))\, ds + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, z^*(s))\, ds.$$

$$z_*(r) \leq \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1))r + \int_0^r \mathscr{H}(s, z^*(s))\, ds + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, z^*(s))\, ds. \tag{3.2}$$

Then the function z^* and z_* are called a pair of upper and lower solutions for the equation -(1.1).

Theorem 3.3 *Assume that [C1] holds. Let z_* and z^* are respectively lower and upper solutions of (1.1), then the problem (1.1) exists at least one solution $z(r) \in B$ satisfying $z_*(r) \leq z(r) \leq z^*(r)$, $r \in [0, 1]$.*

Proof: Let, $U = \{z \in K : z_*(r) \leq z(r) \leq z^*(r), r \in [0, 1]\}$. As $U \subset K$ and U is a nonempty bounded, closed and convex subset of K. Then by lemma - 3.1, $\mathscr{P} : U \to K$ is completely continuous. Also by lemma - 3.2, $\mathscr{Q} : U \to K$ is contraction. Now, we show that if $u(r), v(r) \in U$ then $(\mathscr{P}u)(r) + (\mathscr{Q}v)(r) \in U$. For any $u(r), v(r) \in U$, we have $z_*(r) \leq u(r), v(r) \leq z^*(r)$, then

$$\begin{aligned}
(\mathscr{P}u)(r) + (\mathscr{Q}v)(r) &= \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1)r + \int_0^r \mathscr{H}(s, v(s))\, ds \\
&+ \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, u(s))\, ds, \\
&\leq \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1)r + \int_0^r \mathscr{H}(s, z^*(s))\, ds \\
&+ \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, z^*(s))\, ds, \\
&\leq z^*(r).
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
(\mathscr{P}u)(r) + (\mathscr{Q}v)(r) &= \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1))r + \int_0^r \mathscr{H}(s, v(s))\, ds \\
&+ \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, u(s))\, ds, \\
&\geq \zeta_1 + (\zeta_2 - \mathscr{H}(0, \zeta_1))r + \int_0^r h(s, z_*(s))\, ds \\
&+ \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \mathscr{F}(s, z_*(s))\, ds, \\
&\geq z_*(r).
\end{aligned} \tag{3.4}$$

Thus from (3.3) and (3.4), $z_*(r) \leq (\mathscr{P}u)(r) + (\mathscr{Q}v)(r) \leq z^*(r)$ implying that $(\mathscr{P}u)(r) + (\mathscr{Q}v)(r) \in U \subset K$. Hence all condition of the Krasnoselskii fixed point theorem are satisfied. Then there exists a fixed point $w(r) \in K$, $0 < r \leq 1$. Therefore the problem -(1.1) exists at least one solution $z(r) \in K$ and $z_*(r) \leq z(r) \leq z^*(r)$, $r \in [0,1]$.

Corollary 3.4 *Assume that, [C1] and [C2] holds and there exists θ_1, θ_2, θ_3, $\theta_4 > 0$ such that*

$$\theta_1 \leq \mathscr{H}(r,z) \leq \theta_2, \quad (r,z) \in [0,1] \times [0,\infty), \tag{3.5}$$

and

$$\theta_3 \leq \mathscr{F}(r,z) \leq \theta_4, \quad (r,z) \in [0,1] \times [0,\infty). \tag{3.6}$$

Then the problem-(1.1) has at least one solution $z \in B$. Moreover,

$$\begin{aligned} \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1) + \theta_1)r + \theta_3 \frac{r^\mu}{\Gamma(\mu+1)} &\leq z(r) \\ \leq \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1) + \theta_2)r + \theta_4 \frac{r^\mu}{\Gamma(\mu+1)}&. \end{aligned} \tag{3.7}$$

Proof: By the definition of upper and lower control function and given assumption (3.6), we have

$$\theta_3 \leq m[r,z] \leq M[r,z] \leq \theta_4. \tag{3.8}$$

Now, let

$$\begin{aligned} z_*(r) &= \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1) + \theta_1)r + \theta_3 \frac{r^\mu}{\Gamma(\mu+1)}, \\ &= \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1) + \theta_1)r + \theta_3 \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds. \end{aligned}$$

Taking into account (3.5), (3.8) we have

$$\begin{aligned} z_*(r) &= \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1) + \theta_1)r + \frac{\theta_3}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds, \\ &\leq \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1))r + \int_0^r \mathscr{H}(r, z_*(r))\, ds + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} m(s, z_*(s))\, ds. \end{aligned}$$

It is clear that $z_*(r)$ is the lower solution of (1.1)

$$\begin{aligned} z^*(r) &= \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1) + \theta_1)r + \theta_3 \frac{r^\mu}{\Gamma(\mu+1)}, \\ &= \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1)r + \theta_2 \int_0^r ds + \theta_4 \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds. \end{aligned}$$

Taking into account (3.5), (3.8) we have

$$\begin{aligned} z^*(r) &= \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1)r + \theta_1 r + \frac{\theta_3}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds, \\ &\geq \zeta_1 + (\zeta_2 - \mathscr{H}(0,\zeta_1))r + \int_0^r \mathscr{H}(r, z^*(r))\, ds + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} M(s, z^*(s))\, ds. \end{aligned}$$

It is clear that $w^*(r)$ is the upper solution of equation-(1.1). Therefore, an application of theorem-3.3, yields that the problem -(1.1) has at least one solution $w \in B$ and satisfies equation (3.7).

Theorem 3.5 *Assume that [C1] and [C2] holds and*

$$\alpha + \frac{\beta}{\Gamma(\mu+1)} < 1, \tag{3.9}$$

then the problem -(1.1) has a unique solution $z \in K$.

Proof: From theorem-3.3 follows that the problem -(1.1) has at least one solution in B. Hence for uniqueness of solution, we need only to prove that the operator $\varphi : K \to K$ defined in equation- (3.1) is a contraction on B. For all $\xi_1, \xi_2 \in B$, we have

$$\begin{aligned}|(\varphi\xi_1)(r) - (\varphi\xi_2)(r)| &\le |\mathscr{H}(r,\xi_1(r)) - \mathscr{H}(r,\xi_2(r))| \\ &+ \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} |\mathscr{F}(s,\xi_1(s)) - \mathscr{F}(s,\xi_2(s))| \, ds, \\ &\le \alpha\,|\xi_1 - \xi_2| + \beta\,|\xi_1 - \xi_2| \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds, \\ &= \alpha\,|\xi_1 - \xi_2| + \beta\,|\xi_1 - \xi_2| \frac{r^{\mu}}{\Gamma(\mu+1)}. \\ &\le \left(\alpha + \frac{\beta}{\Gamma(\mu+1)}\right) |\xi_1 - \xi_2|.\end{aligned}$$

Thus,

$$\|\varphi(\xi_1) - \varphi(\xi_2)\| \le \left(\alpha + \frac{\beta}{\Gamma(\mu+1)}\right) \|\xi_1 - \xi_2\|.$$

Hence by equation -(3.9), the operator φ is a contraction mapping. Then by contraction mapping principle, we conclude that the problem -(1.1) has a unique solution $z(r) \in B$.

References

[1] S. Abbas, Existence of solutions to fractional order ordinary and delay differential equations and applications, Electronic Journal of Differential Equations, (9) (2011), 1?1.

[2] A. Ardjouni and A. Djoudi, *Positive solutions for first-order nonlinear Caputo-Hadamard fractional differential equations*, Malaya Journal of Matematik, **8**, (2) (2020), 383–388.

[3] H. Boulares, A. Ardjouni and Y. Laskri, *Positive solutions for nonlinear fractional differential equations*, Positivity, **21** (2017), 1201?1212.

[4] A. Kilbas, H. Srivastava and J. Nieto, *Theory and Applicational Differential Equations*, Elsevier, Amsterdam, 2006.

[5] V.Lakshmikantham and A.Vatsala, *Basic theory of fractional differential equations*, Nonlinear Analysis: TMA, 69, (8) (2008), 2677–2682.

[6] V. Lakshmikantham and A. Vatsala, *Theory of fractional differential inequalities and applications*, Communications in Applied Analysis, Vol-II (2007), 395–402.

[7] M. Li, J.Sun and Y. Zhao, *Existence of positive solution for BVP of nonlinear fractional differential equation with integral boundary conditions*, Advances in Difference Equations (2020) 177.

[8] M. Matar, *On existence of positive solution for initial value problem of nonlinear fractional differential equations of order* $1 < \alpha \leq 2$, Acta Math. Univ. Comen. **84** (1)(2015), 51–57.

[9] K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, (1993).

[10] I. Podlubny, *Fractional Differential Equations, in: Mathematics in Science and Engineering*, Academic Press, New York, London, Toronto, (1999).

[11] S. Zhang, *The existence of a positive solution for a nonlinear fractional differential equation*. J. Math. Anal. Appl., 252 (2000), 804–812.

[12] D. R. Smart, *Fixed point theorem*, Cambridge Tracts in Mathematics, no. 66,Cambridge University Press, London-New York, (1974).

Principal
Jawahar Arts, Science & Commerce College
Anadur, Tal. Tuljapur, Dist. Osmanabad.